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An Analysis of Inertial Seismometer-Galvanometer Combinations

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AN ANALYSIS OF INERTIAL SEISMOMETER-GALVANOMETER COMBINATIONS

Section 1

INTRODUCTION

1.1 Background

Previous interest at the National Bureau of Standards in seismic instrumentation seems to be limited entirely to the work of the late Dr. Frank Wenner whose realization of the interaction between an electromagnetic inertial instrument and its associated galvanometer led him to develop his horizontal instrument.

In the fall of 1954 the authors became interested in seismic instrumentation with particular interest in a vertical component device having the least possible instrumental noise, consistent with minimum weight, with a pass-band of roughly 1 to 5 cps and designed for field use by relatively unskilled personnel.

The study began with a theoretical and practical examination of the system composed of a Benioff Variable Reluctance Seismometer and a d'Arsonval galvanometer and phototube amplifier. Alternative detection methods were considered. It became apparent that an inertial instrument with an electromagnetic sensing element has an inherent instrumental noise level which, at least in principle, is that due to the thermal agitation of the inertial mass. It also became apparent that the instrumental noise of a galvanometer can be as low as the thermal agitation of its coil. We therefore concluded that the combination of an electromagnetic pickup and galvanometer has, in principle, a minimum noise level which may perhaps be equalled by other position sensing arrangements but no other arrangement can exceed this combination as to ultimate sensitivity.

1.2 Scope

In order properly to understand the scope and limitations of a theoretical treatment of the equations of motion for seismograph-galvanometer combinations and the amplitude-frequency response curves derived from them, the authors found it necessary to start from first principles. Where assumptions have been made in the development there has been an attempt to state them explicitly.

The reader will note that there is practically no reference to the work of others in the main body of this report. The authors are not so naive as to believe that all, or even most, of the conclusions in this report are new. The work was deliberately done without reference to the literature in the hope that: 1) an independent development would result in a more complete understanding on our part than would result from a study of the work of others, and 2) a fresh treatment, unbiased by the lines of attack and assumptions used by others, might reveal relations not previously noted. An obvious disadvantage is that another system of nomenclature, more or less unrelated to the many already in the literature, is introduced. However, an attempt has been made to introduce only symbols having physical significance and which are measurable, at least in principle. The reader should not infer that our conclusions are, in all cases, new to the art or that we are attempting to claim credit for the work of others.

We have collected all the symbols used in the text together with their definitions as the first part of Section 10. The second part of that section gives a few references to pertinent previous work with lists of symbols used and their equivalents in our notation, where a correspondence was found.

It was tempting to include a discussion of each reference as it applied to, or was confirmed by, this report, however such a development would have increased the length of this report excessively and was therefore abandoned.

1.3 Introductory Details

Equations have been numbered using a "Dewey decimal system", the first digit signifying the section and the second the subsection in which the equation is developed.

Throughout the course of the work we have noted relations which could be used for evaluating specific instruments or which seemed especially useful to the designer of a seismic instrument or system.

Despite the length of this report no attempt has been made to exhaust the subject as there remain many interesting facets to be explored.

In numerical calculations based on the equations in this report the authors follow the Système International (MKS) since the practical electrical units may be used with it directly. Use of the CGS system results in the appearance of various conversion factors in unobvious places.

Section 2

GENERAL EQUATIONS OF MOTION OF AN ELECTROMAGNETIC INERTIAL SEISMOMETER

2.1 Dynamical Theory

2.1.1 Mechanics

A general formulation of Newtonian mechanics was made by Lagrange. A modern treatment may be found in Whittaker. Clerk Maxwell gives a statement in physicist's language.

A mechanical system can be regarded as a black box with push rods each of which can move along a straight line. The number of push rods is equal to the number of independent variables required to define the configuration of the system. The equations of motion define relations between the positions and velocities of the rods, and the forces applied to them, without regard to the mechanism inside the black box.

Let q_k denote the position, $\dot{q}_k$ the velocity, and Q_k the external force applied to the push rod numbered k. If the configuration at any time is specified by the positions of the push rods, the motion can be specified in terms of the velocities $\dot{q}_k$. In Newtonian mechanics, the kinetic energy T of the system is the sum of terms each of which contains the square of a velocity $\dot{q}_k^2$, or a product of a pair of velocities $\dot{q}_i\dot{q}_j$, with a coefficient which may depend on the coordinates. There may exist within the black box a potential energy V which is a function of the coordinates. A function $\mathcal{L}$ (known as the Lagrangian function) is defined as the difference between the kinetic and potential energy, that is:

$$\mathcal{L} = T - V \qquad (2.1.1)$$

Then the general equation of motion is,

$$Q_k = \frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{q}_k}\right) - \frac{\delta\mathcal{L}}{\delta q_k} \quad \text{or} \qquad (2.1.2)$$

$$Q_k = \frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q}_k}\right) - \frac{\delta T}{\delta q_k} + \frac{\delta V}{\delta q_k} .$$

As developed by Lagrange these equations apply to systems which are conservative. Energy losses, such as those due to electrical resistance, viscosity, hysteresis, etc., may be handled by introducing retarding forces with the external forces.

The Lagrangian equations of motion do not apply directly to systems which contain internal constraints, in which the configuration is not uniquely defined by the positions of the push rods accessible from the outside. A familiar example is that of a ball which is constrained to roll on a table, without sliding. If the ball is rolled around a closed path, it may be returned to the starting point in an orientation differing from that at the start. Additional coordinates will be required to specify the configuration. The constraints impose relations between the velocities consistent with the linkage. For example, the condition that the ball must rotate as it moves so that there is no relative motion at the point of contact, is expressed by two relations between translational and rotational velocities.

The extension of Lagrange's equations to such systems is discussed by Whittaker in Article 87, Chapter VIII, essentially as follows:

Each constraint within the system may be represented by a relation of the form:

$$A_1\delta q_1 + A_2\delta q_2 \dots + A_t\delta t = 0 \qquad (2.1.3)$$

or

$$A_1\dot{q}_1 + A_2\dot{q}_2 \dots + A_t = 0 \qquad (2.1.3a)$$

The constraint may be associated with a set of additional forces Q'_1, Q'_2 which compel the system to move in a direction consistent with the relations between velocities. These forces do no work in an instantaneous displacement consistent with the conditions of the constraint. That is:

$$Q'_1\delta q_1 + Q'_2\delta q_2 \dots + Q'_n\delta q_n = 0 \qquad (2.1.4)$$

for an instantaneous displacement ($\delta t = 0$) such that

$$A_1\delta q_1 + A_2\delta q_2 \dots + A_n\delta q_n = 0 \qquad (2.1.5)$$

These equations are consistent if

$$\frac{Q'_1}{A_1} = \frac{Q'_2}{A_2} = \dots = \frac{Q'_n}{A_n} \qquad (2.1.6)$$

A sufficient set of equations of motion for the system consists of the Lagrangian equations with the addition of the set of forces Q'_1, Q'_2, etc. Equations 2.1.4, 2.1.5 and 2.1.6 represent the conditions on the displacements and forces associated with each constraint.

2.1.2 Electrodynamics The extension of dynamical theory to electromagnetism.

Maxwell showed that the information then known about electricity and the magnetic fields due to currents could be brought into the scope of the dynamical theory already developed for mechanics. The energy in electrostatic fields is added to the mechanical potential energy so that

$$V = V_{mech} + V_{electrostatic} \qquad (2.1.7)$$

$$V_{electrostatic} = \int_{\text{all space}} \frac{\mathcal{D}.\mathcal{E}}{2}\, d\,(\text{Vol}) \qquad (2.1.8)$$

where $\mathcal{E}$ is the electric field strength and $\mathcal{D}$ the electric displacement.

The energy of a magnetic field is added to the mechanical kinetic energy, that is:

$$T = T_{mech} + T_{mag} \qquad (2.1.9)$$

$$T_{mag} = \int_{\text{all space}} \frac{\mathcal{B}.H}{8\pi}\, d\,(\text{Vol}) \qquad (2.1.10)$$

where H is the intensity of the magnetic field and $\mathcal{B}$ the magnetic induction.

In setting up his field equations, Maxwell regarded the quantity of electricity as a coordinate, the electrical current as a velocity, the electromotive force as a force, and the magnetic vector potential as the "electrokinetic momentum".

When the attempt was made to apply these general methods to the problem of the variable reluctance transducer difficulties were encountered. The similarities between electric and magnetic fields may trap one into considering the field of a permanent magnet as the gradient of a scalar potential function, and the energy of a permanent magnet as potential energy, and treating the forces in the magnetic gap on that basis. The inductive effects of a current were clearly kinetic in nature, but in analyzing this problem, the terms which gave rise to the motor and generator constants were hybrid, neither kinetic nor potential, and had no part in the Lagrangian theory.

Maxwell considered this matter at length and pointed out that the magnetic field of a permanent magnet can be duplicated by a suitable distribution of electrical current. He then stated that a consistent theory could not be obtained unless the energy of all magnetic fields, whether due to currents or to permanent magnets was regarded as a portion of the kinetic energy.

The electric current which simulates a permanent magnet differs from other electric currents in the pick-up since the magnitude is not subject to arbitrary independent variation by application of an external emf. It therefore corresponds to the angular velocity of a rolling ball which cannot be varied independently of the translational velocity. It must be treated as an internal constraint of the system, by methods such as those outlined above.

2.2 Choice of Coordinates

Lagrange attempted to set up equations which were valid regardless of the parameters chosen as coordinates. That the end result is independent of the choice of coordinates constitutes a partial check of the method. However, the coordinates must be independent and must be sufficient in number completely to define the configuration of the system. It is easy to overlook internal constraints, which are associated with coordinates in excess of the number of degrees of freedom.

There remains considerable freedom in the choice of the coordinates to be used in the general equations. Here the coordinates will be chosen as to lead to equations in terms of quantities which are measurable, at least in principle.

At least four coordinates with associated velocities, momenta and forces are required to describe the electromagnetic seismometer.

2.2.1 The coordinate of earth motion

The seismometer as a whole moves as a result of the motion of its support. In use this motion should be that of the earth; in calibration it may be that of a shake table. For the vertical component instrument the coordinate y will be the displacement (positive upward) of the (rigid) frame of the instrument with respect to a stationary point. As a result of the change of y there will be a change in the gravitational potential energy V_{grav}. The mechanical kinetic energy of the system will depend in part on the earth velocity $\dot{y}$.

The force $Q_y = Y$ is the force (positive upward) exerted on the instrument frame by its support.

2.2.2 The coordinate of bob motion

The mechanical coordinate x will be taken as the linear displacement of a point on the principal seismic mass or bob with respect to the frame of the instrument, parallel to the motional axis of the instrument. For a magnetically symmetrical instrument it will be measured from the point of symmetry. In a vertical component instrument it will be taken as positive if the mass is displaced upward. The force Q_x acts on the mass along the axis of the instrument. It is taken as positive in the direction to increase x. In addition to a retarding force $-r_x\dot{x}$ there may be forces X exerted on the bob by auxiliary calibrating devices, or by extraneous disturbances. Hence $Q_x = X - r_x\dot{x}$.

It is assumed throughout this work that r_x is a constant, i.e. that retarding forces are proportional to the velocity. A small hysteresis loss will be replaced with a linear retarding force which will produce an equivalent energy loss, ignoring the high order effects.

2.2.3 The electrical coordinate

In a coil connected to an external circuit the electrical coordinate will be taken as the quantity of electricity. The current I_1 will be the generalized velocity and the electromotive force E_1 will be the generalized force. As indicated by the names this is the conventional choice for the electrical coordinate.

With the external emf E_1 will be included a term $- R_c I_1$ to represent the effect of losses due to the electrical resistance of the coils.

As stated above, Maxwell identifies the vector potential of the magnetic induction associated with a current as its "electrokinetic momentum". When integrated over a coil its value is $n\Phi$ where n is the number of turns and Φ the magnetic flux linked with the coil.

2.2.4 The magnetic coordinate

As mentioned before, the external magnetic field of the permanent magnet may be simulated by an electric current. For the sake of symmetry this "phantom" current I_o will be assumed to flow in a simulated coil of the same number of turns n as the real coil associated with the electrical coordinate, Sec. 2.2.3. The electromagnetic momentum of this circuit will be $n\Phi_o$.

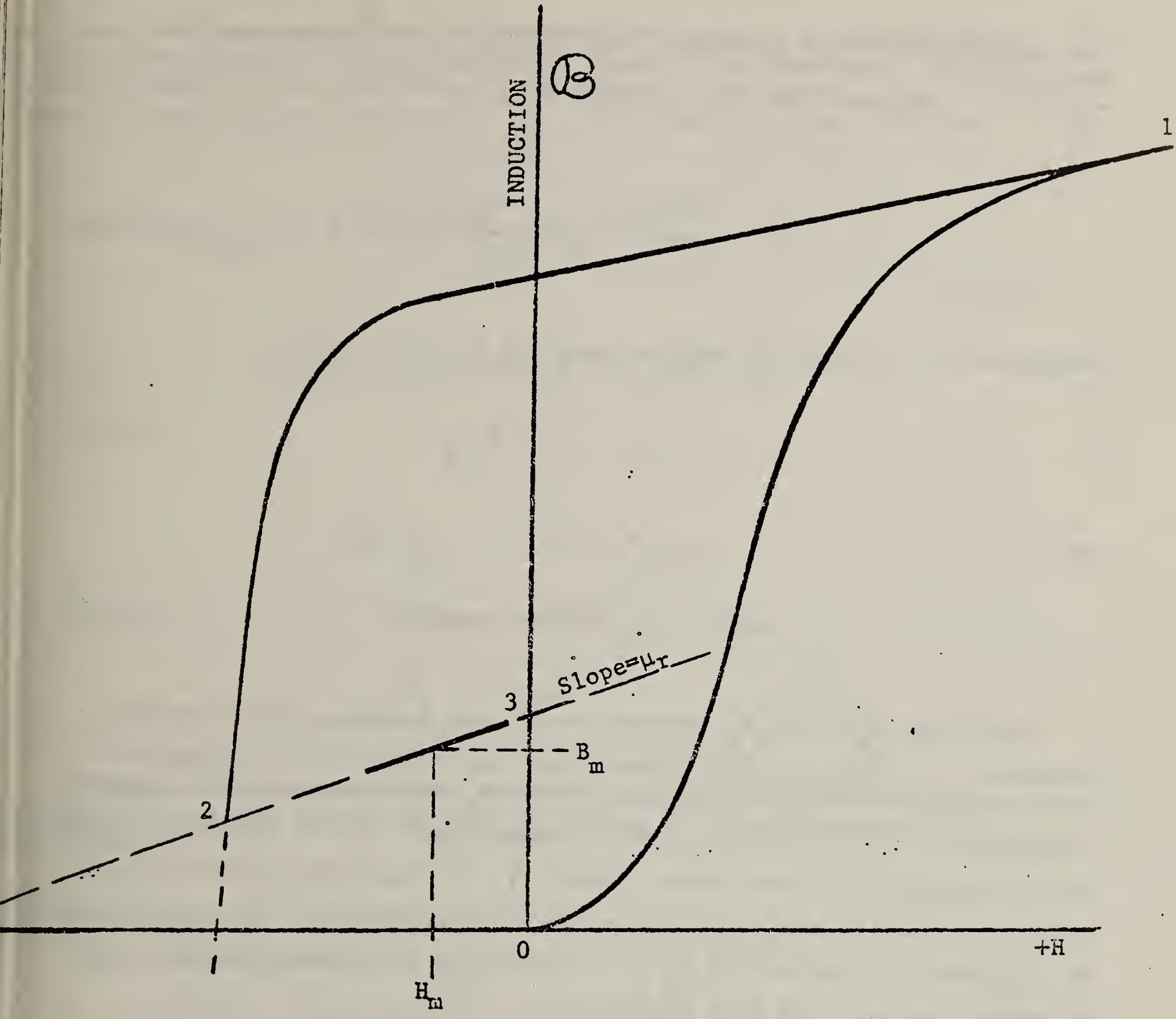
2.3 Condition of Constraint: Magnet

The magnetic coordinate differs from the electrical coordinate in that there is no "push-rod" by which an arbitrary displacement can be produced. In this respect it resembles the mechanical coordinates which are associated with constraints.

The magnetic constraint may be found in the B-H curve of Fig. 2.1 which defines the relation between the induction and field in an element of a magnet. During the process of charging, the magnetic material is carried from the demagnetized state 0, along the curve 01. In the course of the stabilizing process a reverse field is applied to carry it to the point 2, from which it "springs back" along the line $\overline{23}$. The operating region is shown as the solid portion of the line. The operating curve is really a very narrow loop, but can be represented for our purpose as a segment of the straight line $\overline{23}$ the equation of which is:

$$\frac{B_m}{\mu_r} - (\overline{40}) - H_m = 0 \tag{2.3.1}$$

where B_m is the induction and H_m is the magnetizing force in the magnet. The slope $\frac{dB_m}{dH_m}$ of this line is the "reversible permeability" μ_r of the magnetic material.



MAGNETIZING FORCE

Figure 2.1

Let the permanent magnet be simulated by an electromagnet having a coil wound on a core of the same external dimensions as the magnet. For convenience, this coil will be taken as having the same number of turns n as the coil connected to the external circuit. Let ℓ_m be the length and A_m the cross-sectional area of the permanent magnet, μ_o its permeability, and I_o be the current in the coil, so that the mean magnetomotive force per unit length of the magnet will be $\frac{4\pi n I_o}{\ell_m} + H_m$ and the flux will be given by

$$\frac{4\pi n I_o}{\ell_m} + H_m = \frac{\Phi_o}{A_m \mu_o} \qquad (2.3.2)$$

Comparison with eq. 2.3.1 suggests setting

$$\mu_o = \mu_r \; ; \quad \Phi_o = B_m A_m \qquad (2.3.3)$$

and

$$4\pi n I_o = \ell_m \, (\overline{40}), \text{ a constant} \qquad (2.3.4)$$

The flux Φ_o and H_m depend on all the currents and on the displacement x. Equation 2.3.4 shows that the simulating electromagnet must have associated with it a potential E_o which varies with all the currents and displacements in such a manner that I_o is constant. This is the condition of constraint. Since I_o is the only current in eq. 2.3.4 E_o will be the only supplementary emf required. The significance of E_o and μ_o are buried deep in the theory of ferromagnetism, which is beyond the scope of this discussion.

This mathematical artifice may be applied to all the materials in the magnetic structure so that a linear relation between B & H may be said to hold. The assumption of zero hysteresis would be justified in a practical case if the user of a device observed negligible damping chargeable to the magnetic material.

2.4 The Lagrangian Function

2.4.1 Mechanical kinetic energy

The velocity of any elementary mass dm of the seismometer may be divided into an axial and a transverse component. The axial velocity is the sum of the earth velocity $\dot{y}$ and some fraction, a, of the velocity $\dot{x}$ of the bob relative to the instrument frame. The transverse velocity is some fraction, b, of the velocity of the bob. In general, the coefficients a and b are functions of x. The kinetic energy is then

$$T_{mech} = \int \left[\frac{(a\dot{x} + \dot{y})^2}{2} + \frac{(b\dot{x})^2}{2} \right] dm$$

$$= \frac{\dot{x}^2}{2} \int (a^2 + b^2)\, dm + \dot{x}\dot{y} \int a dm + \frac{\dot{y}^2}{2} \int dm \tag{2.4.1}$$

$$\text{Let } M_x = \int (a^2 + b^2)\, dm$$

$$M_{xy} = \int a dm \tag{2.4.2}$$

$$M_y = \int dm.$$

An exact evaluation of these integrals requires a detailed study of each piece of the instrument and the nature of its attachment to all other pieces[1].

Then

$$T_{mech} = \frac{M_x \dot{x}^2}{2} + M_{xy}\dot{x}\dot{y} + \frac{M_y \dot{y}^2}{2} \tag{2.4.3}$$

2.4.2 Electrokinetic and electropotential energy

The energy of the magnetic field is given by the integral of eq. 2.1.10, over all space. In terms of the currents it can be put in the form:

---

[1] The ratio of M_{xy} to M_x enters the equations for instrument performance. It may be made very nearly unity by careful design. This ratio has been accounted for in the work of others by introducing a concept termed "the reduced length".

$$T_{mag} = \frac{1}{2} W_{oo} I_o^2 + W_{o1} I_o I_1 + \frac{1}{2} W_{11} I_1^2 \ . \qquad (2.4.4)$$

Where W_{oo}, W_{o1} and W_{11} depend on the geometry of the magnetic circuit and therefore may be functions of the coordinates.

An alternative expression is obtained by noting that the electromagnetic momenta are given by

$$n\Phi_o = \frac{\delta T_{mag}}{\delta I_o} = W_{oo} I_o + W_{o1} I_1 \qquad (2.4.5)$$

$$n\Phi_1 = \frac{\delta T_{mag}}{\delta I_1} = W_{o1} I_o + W_{11} I_1 \qquad (2.4.6)$$

With these substitutions, the electromagnetic energy is given by:

$$T_{mag} = \frac{1}{2} n\Phi_o I_o + \frac{1}{2} n\Phi_1 I_1 \qquad (2.4.7)$$

Here the energy is expressed in terms of the currents in the coils and the flux linking each of them.

Electrostatic energy, such as that associated with capacities in the transducer, is potential in form and ignored in this development.

2.4.3 Gravitational potential energy

In the vertical component seismometer the change δV_{grav} in gravitational potential energy due to small displacements δx and δy is equal to the integral over the mass of the axial displacement of the mass due to these displacements.

$$\delta V_{grav} = \int (a\delta x + \delta y)\ g dm. \tag{2.4.8}$$

From eq. 2.4.1 this may be integrated over the mass to give:

$$\delta V_{grav} = gM_{xy}\delta x + gM_{y}\delta y \tag{2.4.9}$$

Integration with respect to x and y gives:

$$V_{grav} = \int^{x} gM_{xy} dx + \int^{y} gM_{y} dy. \tag{2.4.10}$$

The reader will note that here g is taken to be the constant upward acceleration of the whole instrument required to replace the downward attraction due to gravity.

2.4.4 Mechanical potential energy

The energy V_s is that stored in the elastic distortion of the primary and other springs. Since the springs connect, in one way or another, the bob and the frame of the instrument, their potential energy V_s is a function of x only. If they are linear the function is a quadratic. If they are non-linear, other powers than the second will be required to represent the potential energy.

In general, a practical instrument will respond to transverse and rotational motions. It will also have multiple and distributed masses and elastic elements. In a good mechanical design these effects are made negligible. Their consideration at this time would lead to complications of the analysis which would obscure the points to be developed. Therefore we will restrict this discussion to a uniaxial device with a minimum number of degrees of freedom.

It should be repeated that the tractive forces of the permanent magnet enter the Lagrangian with the electrokinetic energy. These forces are analogous to the centrifugal forces on the bobs of a governor or on the arms of a spinning skater. They may resemble forces derived from the potential energy in some respects, but must be treated as kinetic in nature. That is, the negative stiffness introduced into an instrument by a magnetic structure does not appear in the potential energy terms of the Lagrangian function.

2.5 The General Equations of Motion

It is now possible to write the Lagrangian function of the Vertical Seismometer.

$$\mathcal{L} = \frac{1}{2} M_x \dot{x}^2 + M_{xy} \dot{x}\dot{y} + \frac{1}{2} M_y \dot{y}^2$$

$$+ \frac{1}{2} W_{oo} I_o^2 + W_{o1} I_1 I_o + \frac{1}{2} W_{11} I_1^2 \qquad (2.5.1)$$

$$- \int^x M_{xy} g dx - \int^y M_y g dy - V_s$$

Note that the Lagrangian function as written here contains cross product terms between the mechanical velocities $\dot{x}$ and $\dot{y}$ and between the currents I_o and I_1, but no cross product between a mechanical velocity and an electrical current. Cross products between mechanical velocities and electrical currents are associated with such subtleties as the Hall effect, and the gyromagnetic effect. The first applies to the current distribution within conductors. The second relates the change in magnetization to a torque on the coil core. Such effects are ignored in this development.

From eq. 2.3.4 the magnetic constraint is given by

$$I_o = \text{a constant.} \qquad (2.5.2)$$

The general equations of motion are obtained by performing the Lagrangian operation, eq. 2.1.2, on the Lagrangian function, eq. 2.5.1 to give:

$$Y = \frac{d}{dt} (M_{xy} \dot{x} + M_y \dot{y}) + g M_y \qquad (2.5.3)$$

$$X - r_x \dot{x} = \frac{d}{dt}(M_x \dot{x} + M_{xy} \dot{y}) + gM_{xy} + \frac{dV_s}{dx} - \frac{1}{2} \dot{x}^2 \frac{dM_x}{dx} - \dot{x}\dot{y} \frac{dM_{xy}}{dx} - \frac{1}{2} I_o^2 \frac{dW_{oo}}{dx} - I_o I_1 \frac{dW_{o1}}{dx} - \frac{1}{2} I_1^2 \frac{dW_{11}}{dx} \quad (2.5.4)$$

$$E_1 - R_c I_1 = \frac{d}{dt}(W_{o1} I_o + W_{11} I_1) \quad (2.5.5)$$

$$E_o = \frac{d}{dt}(W_{oo} I_o + W_{o1} I_1) \quad (2.5.6)$$

2.6 The Linearized Equations of Motion

The following manipulations will make these equations easier to handle:

First: Let the rest point $x = x_o$ be defined as the equilibrium position of the bob in the absence of earth motion $\dot{y}$, external forces X and emf E_1. Equation 2.5.4 then gives

$$0 = gM_{xy} + \frac{dV_s(x_o)}{dx} - \frac{1}{2} I_o^2 \frac{dW_{oo}(x_o)}{dx} \quad (2.6.1)$$

The rest point may be regarded as fixed in considering the response of the instrument to a signal. It will drift relatively slowly as the ambient conditions change over intervals of hours or days. For small excursions in the neighborhood of the rest point, squares, products, and terms of higher power in the excursion, $x - x_o$, the velocities $\dot{x}$ and $\dot{y}$ and the current I_1 will be assumed small with respect to the linear terms in these variables.

Second: The quantities which are functions of x may be developed as Taylor series in the neighborhood of the rest point x_o. For example:

$$V_s(x) = V_s(x_o) + (x - x_o)\frac{dV_s(x_o)}{dx} + \frac{(x - x_o)^2}{2.}\frac{d^2V_s(x_o)}{dx^2} \quad \text{etc.} \tag{2.6.2}$$

For the linearized equations terms as high as the second order will be retained for V_s and W_{oo}, and terms of the first order for M_{xy}, W_{o1} and W_{11}.

Third: In performing the indicated differentiation with respect to time, note that $\frac{dW_{oo}}{dt} = \dot{x}\frac{dW_{oo}}{dx}$ and $\frac{dW_{o1}}{dt} = \dot{x}\frac{dW_{o1}}{dx}$.

Equations 2.5.3 through 2.5.6 now reduce to:

$$Y = M_y\ddot{y} + M_{xy}\ddot{x} + M_y g \tag{2.6.3}$$

$$\begin{aligned} X - M_{xy}\ddot{y} = {} & M_{xy}g + \frac{dV_s(x_o)}{dx} - \frac{1}{2}I_o^2\frac{dW_{oo}(x_o)}{dx} + M_x\ddot{x} + r_x\dot{x} \\ & + \left[\frac{d^2V_s}{dx^2} - \frac{1}{2}I_o^2\frac{d^2W_{oo}}{dx^2} + \frac{dM_{xy}}{dx}g\right](x - x_o) - I_o\frac{dW_{o1}}{dx}I_1 \end{aligned} \tag{2.6.4}$$

$$E_1 = I_o\frac{dW_{o1}}{dx}\dot{x} + W_{11}\dot{I}_1 + R_c I_1 \tag{2.6.5}$$

$$E_o = I_o \frac{dW_{oo}}{dx} \dot{x} + W_{o1} \dot{I}_1 \qquad (2.6.6)$$

Equation 2.6.3 gives the force applied to the instrument as a whole. It would be used in considering the flexure of a pier, in shake table calibrations, or tests on an isolating block ("stable table").

Equation 2.6.6 defines the internal behavior of the permanent magnet, or the back emf in the simulating coil.

Equations 2.6.4 and 2.6.5 define the relation between earth motion, bob displacement and current in the seismometer.

Fourth: It is now possible to identify certain coefficients in eqs. 2.6.4 and 2.6.5 as the physical quantities S, G and L:

$$S = \frac{d^2V_s}{dx^2} - \frac{1}{2} I_o^2 \frac{d^2W_{oo}}{dx^2} + \frac{dM_{xy}}{dx} g \qquad (2.6.7)$$

is the resultant spring stiffness of the system,

$$G = I_o \frac{dW_{o1}}{dx} \qquad (2.6.8)$$

is the "motor constant" in eq. 2.6.4 and the "generator constant" in eq. 2.6.6 and

$$L = W_{11} \qquad (2.6.9)$$

is the coil inductance.

These quantities will be regarded as constants over the period of any signal to which the seismometer responds. They are functions of the rest point x_o and therefore change with any drift of the rest point.

After subtracting eq. 2.6.1 from 2.6.4 and substituting S, G and L, eqs. 2.6.4 and 2.6.5 reduce to:

$$X - M_{xy}\ddot{y} = M_x\ddot{x} + r_x\dot{x} + S\,(x - x_o) - GI_1 \qquad (2.6.10)$$

$$E_1 = G\dot{x} + L\dot{I}_1 + R_cI_1\ . \qquad (2.6.11)$$

2.7 Philosophical Notes and Interpretations

The identification of the coefficient W_{11} as the self inductance of the coil suggests an interpretation of W_{oo} and W_{o1}. The coefficient W_{oo} may be interpreted as the self inductance of the coil simulating the magnet. The coefficient W_{o1} may be interpreted as the mutual inductance of these two coils. These interpretations are consistent with eqs. 2.4.5 and 2.4.6.

The motor "constant", G in eq. 2.6.10 is equal to the generator "constant" G in eq. 2.6.11, but enters a term of opposite sign. In the electrostatic transducer the corresponding terms are of the same sign. The difference is associated with the difference in the way electrostatic and electromagnetic energies enter the Lagrangian function.

Electrostatic energy is considered as potential energy, and is a function of the coordinates, conventionally displacement and charge. The motor constant would come from differentiation of the interaction energy with respect to displacement, and the generator constant from differentiation with respect to charge. Both are of the same sign.

As emphasized above, the electromagnetic energy enters the Lagrangian with the kinetic energy. The motor constant comes from a differentiation with respect to x, and has a negative sign. The generator constant comes from a differentiation with respect to the current I and has a positive sign.

2.8 Extension to Two Coil Systems

2.8.1 General

In a multicoil instrument the coils may be divided into groups connected into separate electrical circuits. The division may be done in several ways. (a) In a detailed analysis of a symmetrical magnetic circuit it is necessary to consider the upper and lower cores and their coils separately. (b) In service, there is the option of connecting coils on the upper and lower cores in series, or in parallel. The relative merits of these connections must be considered. (c) Use has been made of a damping circuit electrically distinct from the galvanometer connection. (d) There is a possible method of reciprocity calibration, based on measurements in which the coils are divided into two groups which, for convenience, may be made as nearly alike as possible.

2.8.2 Equations of motion

For such arrangements the Lagrangian function may be set up as before, with the obvious additional terms:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} M_x \dot{x}^2 + M_{xy} \dot{x}\dot{y} + \frac{1}{2} M_y \dot{y}^2 \\ &+ \frac{1}{2} W_{oo} I_o^2 + W_{o1} I_o I_1 + W_{02} I_o I_2 \\ &+ \frac{1}{2} W_{11} I_1^2 + W_{12} I_1 I_2 + \frac{1}{2} W_{22} I_2^2 \\ &- \int^x M_{xy} g\,dx - \int^y M_y g\,dy - V_s\end{aligned} \qquad (2.8.1)$$

The following auxiliary quantities analogous to those of section 2.6 may be defined:

$S = \frac{d^2V_s}{dx^2} - \frac{1}{2} I_o^2 \frac{d^2W_{oo}}{dx^2} + \frac{dM_{xy}}{dx} g$ is the resultant stiffness as before,

$G_1 = I_o \frac{dW_{o1}}{dx}$ is the motor-generator constant of coil 1

$G_2 = I_o \frac{dW_{o2}}{dx}$ is the motor-generator constant of coil 2,

$L_1 = W_{11}$ is the self-inductance of coil 1 as before,

$L_2 = W_{22}$ is the self-inductance of coil 2,

$M_{12} = W_{12}$ is the mutual inductance between coil 1 and coil 2.

Equations 2.4.5 and 2.4.6 relating the fluxes Φ_o , Φ_1 , Φ_2 in each core with the currents I_o , I_1 , I_2 become:

$$n\Phi_o = W_{oo}I_o + W_{o1}I_1 + W_{o2}I_2 \tag{2.8.2}$$

$$n\Phi_1 = W_{o1}I_o + W_{11}I_1 + W_{12}I_2 \tag{2.8.3}$$

$$n\Phi_2 = W_{o2}I_o + W_{12}I_1 + W_{22}I_2 \tag{2.8.4}$$

The linearized equations of motion become:

$$Y = M_{xy}\ddot{x} + M_y\ddot{y} + gM_y \tag{2.8.5}$$

$$X - M_{xy}\ddot{y} = M_x\ddot{x} + r_x\dot{x} + S\ (x - x_o) - G_1I_1 - G_2I_2 \tag{2.8.6}$$

$$E_1 = G_1\dot{x} + L_1\dot{I}_1 + R_1I_1 + M_{12}\dot{I}_2 \tag{2.8.7}$$

$$E_2 = G_2\dot{x} + M_{12}\dot{I}_1 + L_2\dot{I}_2 + R_2 I_2 \qquad (2.8.8)$$

$$E_o = \frac{dW_{oo}}{dx} I_o\dot{x} + W_{o1}\dot{I}_1 + W_{o2}\dot{I}_2 \qquad (2.8.9)$$

2.8.3 Reciprocity calibration using the basic instrument

Suppose the coils are divided into two groups as symmetrically as possible. Then $G_1 \sim G_2$ and $L_1 \sim L_2$. Let the mutual inductance be measured with the mass blocked, so that $\dot{x} = 0$, and with sinusoidal excitation, as by measuring the voltage E_2 on open circuit ($I_2 = 0$) and current I_1 . Then, from eq. 2.8.8, with $\dot{x} = 0$ and $I_2 = 0$

$$M_{12} = E_2/\dot{I}_1 = E_2/j\omega I_1 \qquad (2.8.10)$$

The experiment is then repeated with the mass free to move. From eqs. 2.8.6 and 2.8.8, with $I_2 = 0$ and $\dot{y} = 0$:

$$0 = (-\omega^2 M_x + j\omega r_x + S)\ (x - x_o) - G_1 I_1' \qquad (2.8.11)$$

$$E_2' = j\omega G_2\ (x - x_o) + j\omega M_{12} I_1' \qquad (2.8.12)$$

whence

$$\frac{E_2'}{j\omega I_1'} = M_{12} + \frac{G_1 G_2}{(-\omega^2 M_x + j\omega r_x + S)} \qquad (2.8.13)$$

or

$$\left|\frac{E_2'}{I_1'} - \frac{E_2}{I_1}\right|^2 = \frac{G_1^2G_2^2}{SM_x\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)^2 + r_x^2} \qquad (2.8.13a)$$

where $\omega_o = \sqrt{S/M_x}$.

Thus from measurements at various frequencies one can compute S/M_x, $\frac{r_x}{\sqrt{SM_x}}$, and $\frac{G_1G_2}{M_x}$. The mass M_x can be computed from weighings. L_1, L_2, R_1 and R_2 may be measured electrically.

If, in service, the two groups be connected in parallel, then the effective motor-generator constant G will very nearly equal $\sqrt{G_1G_2}$ and the inductance L will very nearly equal $\frac{1}{2}\left(\sqrt{L_1L_2} + M_{12}\right)$. This determines all the constants in the equations of motion.

If the two groups are connected in series. the effective motor-generator constant, $G_1 + G_2$, is very nearly $2\sqrt{G_1G_2}$ and the inductance will be approximately $2\left(\sqrt{L_1L_2} + M_{12}\right)$.

If it is inconvenient to divide the two coils into identical groups, the ratio G_1/G_2 is equal to the ratio $\frac{E_1}{E_2}$ of open circuit voltages when the bob is moved by mechanical means, as by a weight lift, or earth motion. Given the product G_1G_2 and the ratio G_1/G_2, the individual quantities can be determined.

2.8.4 Reciprocity calibration using auxiliary calibrating coils

If the instrument be equipped with an auxiliary calibrating coil and magnet then the method of calibration outlined in 2.8.3 is simplified since M_{12} is zero.

As presently used the motor constant of the calibrator coil where one is used is generally given with the coil, rather than determined during the calibration. However for a seismometer on open circuit Eq. 2.8.13 still applies.

2.9 Application to Other Electromagnetic Transducers

Although the equations of motion have been developed with particular reference to the vertical component Benioff Seismometer, the results may be applied to other electromagnetic transducers.

Equations 2.6.10 and 2.6.11 also apply to the d'Arsonval galvanometer, the coordinate x being regarded as an angle and M_x and M_{xy} as moments and products of inertias. If there is no magnetic material in the armature, W_{oo} is a constant. Usually the inductance of the coil is small enough to be neglected. The d'Arsonval galvanometer is considered in detail in Section 4.

In the dynamic loudspeaker, if x is the displacement of the coil, X is the force with which the air column reacts on the coil. Here also W_{oo} is usually constant, and the inductance of the coil may often be neglected.

The treatment of Section 2.8 applies to the differential transformer in which case I_o is the current in the exciting coil. The self inductance of the exciting coil is W_{oo}, the mutual inductances between the exciting coil and the pick-up coils are W_{o1} and W_{o2}.

In some of the above applications the neglect of electropotential energy (Sec. 2.4.2) may not be valid.

Section 3

RESPONSE CHARACTERISTICS OF A SEISMOMETER WITH A RESISTIVE LOAD

3.1 Introduction

The over-all response characteristics of the seismometer are complicated by a number of minor parameters which can obscure an understanding of the behavior of the instrument. For this reason a number of special cases will be discussed. In the special cases non-linearities and viscous damping will be neglected. When a factor which is normally small becomes significant (as in strong resonances) its effect will be noted. A qualitative discussion of the effects of the minor parameters is included in one of the more general cases. In general the mass M_x, relating to the kinetic energy of the bob, differs from M_{xy}, relating to the potential energy of the bob, because the masses of the spring and other suspension members are involved differently for motion and position. In many practical instruments M_x and M_{xy} differ by a fraction of a percent. Here they will be taken as equal and be represented by M. If the distinction need be made, the correction factor M_{xy}/M_x may be applied to the earth motion y. The rest point x_o in the expression $(x - x_o)$ will be suppressed, with the realization that the parameters of the system may be functions of the rest point.

The more important parameters of the seismometer are: the motor-generator constant G, the mass M of the bob, the resultant spring stiffness S, and the inductance L and resistance R of the electrical circuit.

3.2 Seismometer With Resistive Load

The equations of motion, as developed in Section 2.6 (Eq. 2.6.10 and 2.6.11) may be written as:

$$X - M \frac{d^2y}{dt^2} = M \frac{d^2x}{dt^2} + r_x \frac{dx}{dt} + Sx - GI \qquad (3.2.1)$$

$$E = G \frac{dx}{dt} + L \frac{dI}{dt} + RI \qquad (3.2.2)$$

where I is the current in the electrical circuit. It is taken as positive when positive current flows from the external circuit into the instrument at the positive terminal.

L is the inductance of the circuit, including the coil and any external series inductance.

R is the resistance of the circuit, including the coil and any external resistance.

In the general discussion of Section 2, E_1 was defined as the potential difference measured at the terminals of the seismometer coil, and R_c and L were the resistance and inductance of the coil. If the seismometer is connected in series with a pure resistance, (or a resistance and inductance in series), these external impedances can be combined with those of the coil. When the external circuit is closed, then E is the voltage across such zero impedance generators as may be in the external circuit. The other symbols have already been defined.

Before proceeding with the solution of these equations, it will be convenient to introduce the following quantities:

$\omega_o = \sqrt{S/M}$ is the angular frequency of the seismometer on open circuit, as previously defined in paragraph 2.8.3

$f = \omega/\omega_o$ is the normalized frequency

$\rho = R/L\omega_o$ is the ratio of resistive to inductive impedance at the open circuit frequency.

$\gamma = G/\sqrt{LS} = G/\omega_o \sqrt{LM}$ is the normalized motor-generator constant.

$\tau = \omega_o t$ is the normalized time variable. When measured in these units, the open circuit angular frequency of the seismometer is one or the open circuit period is 2π.

$D = d/d\tau = d/\omega_o dt$ is the normalized time derivative operator.

$\rho_x = r_x/M\omega_o$ is the normalized internal damping coefficient.

$f_s^2 = \gamma^2 + 1$; this will be shown to be the "short circuit frequency".

With these substitutions, the equations of motion can be put into normalized form:

$$\frac{X}{S} - D^2y = (D^2 + \rho_x D + 1)\, x - \gamma I \sqrt{L/S} \tag{3.2.3}$$

$$E/\omega_o L = \gamma D x \sqrt{S/L} + (D + \rho)\, I \tag{3.2.4}$$

The quantity L/S is the ratio, in consistent units, of the energy stored in the inductance by unit current, to the energy stored in the spring system by unit displacement. These equations may be solved for the current I by eliminating the bob displacement x, neglecting the mechanical damping, i.e., $\rho_x = 0$, to give:

$$\gamma D\,(D^2y - X/S) + (D^2 + 1)\,\frac{E}{\omega_o \sqrt{LS}} = [D^3 + \rho D^2 + f_s^2\, D + \rho]\, I \sqrt{L/S} \tag{3.2.5}$$

Mathematically, γ and ρ are independent variables and may, at least in principle, take on any values whatsoever, however, as indicated below, the instrumental adjustments available to the user of a vertical component Benioff variable reluctance seismometer affect both parameters.

An increase of tension in the horizontal tension strips increases ω_o and therefore reduces both γ and ρ in the same proportion.

An increase of magnet charge, without other change, will increase γ since the generator constant will be increased and the effective spring stiffness will be decreased. At the same time ρ will be increased since ω_o is decreased.

The user may, within limits, change γ without changing ρ by (a) simultaneously changing the magnet charge and horizontal tension strips to keep the free period of the instrument constant or (b) simultaneously changing the horizontal tension strips and external resistance to keep the ratio of total circuit resistance to free period constant.

There is a minimum value of ρ, nearly independent of coil configuration, which for this instrument is about 1.5 times the free period in seconds. This value of ρ corresponds to zero external resistance and is increased by increasing the external resistance. On the other hand, any value of γ is nearly independent of coil configuration and is completely independent of external resistance.

3.2.1 Steady state response

3.2.1.1 Steady state response to earth motion, shake table.

Let the instrument be connected to a resistive load, and subjected to sinusoidal excitation at angular frequency ω, as by earth motion or by a shake table with displacement y. Let X and E both be zero. Replacing the operator D by jf, Eq. 3.2.5 becomes:

$$\gamma f^3 y = [(f^2 - f_s^2) f - j\rho (f^2 - 1)] I \sqrt{L/S} \qquad (3.2.6)$$

The phase angle ψ_y by which the current leads the earth motion is given by

$$\tan \psi_y = \frac{\rho (f^2 - 1)}{f (f^2 - f_s^2)}$$

The ratio of the absolute magnitudes of the current and the earth displacement is given by

$$\left| \frac{I}{y} \right| = \sqrt{S/L} \frac{\gamma f^3}{\sqrt{f^2 (f^2 - f_s^2)^2 + \rho^2 (f^2 - 1)^2}}$$

The Nyquist diagram, the phase-frequency curve, and the amplitude-frequency curve for the steady state response can be plotted from these equations. These equations have the following properties:

(a) At the open circuit frequency $f = 1$,

$$\frac{I}{y}\sqrt{L/S}\,\Big|_{\omega = \omega_o} = -\frac{1}{\gamma},$$ and the current is 180° out of phase with the earth displacement, or in phase with the acceleration;

(b) At low frequencies $f \ll 1$, $\frac{I}{y}\sqrt{L/S}\,\Big|\xrightarrow[\omega \to 0]{} -j\frac{\gamma}{\rho}f^3$.
That is, as the frequency is decreased the phase angle approaches 270°. Its cotangent approaches $(f_s^2 f/\rho)$;

(c) At the frequency $f_s \equiv \sqrt{\gamma^2 + 1}$,

$$\frac{I}{y}\sqrt{L/S}\,\Big|_{\substack{\omega = \omega_s \\ \text{or } f = f_s}} = j\frac{(\gamma^2 + 1)^{3/2}}{{}_2\rho\delta} = jf_s^3/\rho\gamma$$ the current

is in phase with the velocity;

(d) On the high frequency asymptote, $\frac{I}{y}\sqrt{L/S}\,\Big|\xrightarrow[\omega \to \infty]{} \gamma$
the phase angle approaches zero and its tangent approaches ρ/f.

It should be noted that at high frequencies and at $f = 1$, the seismometer has an output current which is independent of ρ, and therefore of the load resistance. Accordingly, the instrument can be regarded as a high impedance generator, or a constant current source, at these frequencies.

At low frequencies and at $f = f_s$ the output current is inversely proportional to ρ or to the circuit resistance R. At these frequencies the seismometer can be regarded as approximating a low impedance generator, or a constant voltage source.

So far, the mechanical losses have been neglected ($\rho_x = 0$). This approximation is not valid if the electrical losses are comparably small, as occurs when ρ is very small or very large. The general expression for ρ_x and ρ of the same order is messy, but the two limiting cases, $\rho = 0$ and $\rho = \infty$ will illustrate the point.

For the case of short circuited superconducting coils $\rho = 0$, $E = 0$. Equation 3.2.4 then gives:

$$\gamma Dx \sqrt{S/L} + DI = 0 \quad \text{whence} \quad I = - \gamma x \sqrt{S/L}$$

substituting for I in Eq. 3.2.3 then gives:

$$\frac{X}{S} - D^2 y = (D^2 + \rho_x D + 1 + \gamma^2)\, x \qquad (3.2.7)$$

showing a strong resonance at $\omega \simeq \omega_o \sqrt{1 + \gamma^2}$ or $f \simeq f_s$. For this reason f_s is called the "short circuit frequency".

f_s may also be thought of as that frequency at which the spring-mass combination reflects into the electrical circuit as a capacitative reactance (cf. paragraph 3.2.4) which is in series resonance with the electrical inductance of the coil. It is easily measured, cf. paragraph 3.2.1.3(c) and Figs. 4.2 and 4.3.

On open circuit $\rho = \infty$ and therefore the current is zero. Equation 3.2.3, with $I = 0$ shows a strong resonance in the motion of the bob at $f \simeq 1$, or $\omega \simeq \omega_o$, that is x will be large. Then, from 3.2.4, the open circuit voltage E will also be very large at $f \simeq 1$. This behavior is consistent with the definition of ω_o as the open circuit angular frequency.

3.2.1.2 Steady state response to bob force, calibrator

Using an auxiliary coil and magnet, not linked electrically or magnetically to the reluctance transducer, or a light spring and eccentric it is possible to apply a controlled variable force directly to the bob of the seismometer. Such arrangements are used as remote calibrators since the force applied to the bob may be made quite linearly proportional to the current in the auxiliary coil or to the total indicated runout of the eccentric.

For this case y and E are both zero. Assuming sinusoidal excitation the operator D in Eq. 3.2.5 may be replaced by jf to give Eq. 3.2.8, wherein X becomes the product of the motor constant of the calibrator and the calibrating current or the change in force applied by the calibrator spring.

$$\gamma f\ X/S = [f\ (f^2 - f_s^2) - j\rho\ (f^2 - 1)]\ I\sqrt{L/S} \tag{3.2.8}$$

The phase angle by which the seismometer current leads force exerted by the calibrator is given by

$$\tan \psi_X = \frac{\rho\ (f^2 - 1)}{f\ (f^2 - f_s^2)}$$

that is, the phase relations are the same for excitation by a calibrator as for earth motion.

The absolute magnitude of the ratio of the seismometer current and calibrator force is given by

$$\left|\frac{I}{X}\right| = \frac{\gamma . f}{\sqrt{SL\ [f^2\ (f^2 - f_s^2)^2 + \rho^2\ (f^2 - 1)^2]}}$$

From these equations the Nyquist diagram, the phase-frequency curve, and the amplitude-frequency curve for the steady state response can be plotted. These equations have the following properties:

(a) At the open circuit frequency $f = 1$,

$\frac{I}{X}\Big|_{\omega = \omega_o} = \frac{-1}{\gamma\sqrt{SL}}$ and the seismometer current is 180° out of phase with the calibrator force.

(b) At low frequencies $f \ll 1$, $\frac{I}{X}\Big| \xrightarrow[\omega \to 0]{} \frac{-jf\gamma}{\rho\sqrt{SL}}$ and the phase angle approaches 270°.

(c) At the frequency $f = f_s \equiv \sqrt{\gamma^2 + 1}$, $\frac{I}{X}\Big|_{f = f_s} = jf_s/\rho\gamma\sqrt{SL}$.

And the current is in quadrature with the force.

(d) On the high frequency asymptote,

$\frac{I}{X} \Big| \xrightarrow[\omega \to \infty]{} \frac{\gamma}{f^2 \sqrt{SL}} = \frac{G}{f^2 SL}$. The phase angle approaches zero and its tangent approaches ρ/f.

While this treatment has neglected mechanical losses (ρ_x assumed zero) it will be essentially correct at any reasonably small values of ρ_x.

3.2.1.3 Steady state response to electrical excitation

Let the seismometer be excited by a sinusoidal voltage applied in the electrical network, with no ground motion or externally applied forces on the bob. Then, setting $D = jf$, Eq. 3.2.5 takes the form:

$$\frac{E}{I} = R + jf\omega_o L \left(\frac{f_s^2 - f^2}{1 - f^2}\right) \qquad (3.2.9)$$

By inspection, it can be seen that:

(a) At $f \ll 1$, measurement of the applied voltage and resulting current determines the parameter R.

(b) In the neighborhood of $f = 1$ the relative phase of applied voltage and resulting current changes very rapidly.

(c) At $f = f_s$ the applied voltage and resulting current are in phase.

(d) At $f \gg f_s$ the ratio of the applied voltage and resulting current is dominated by the inductance.

The above treatment assumes $\rho_x = 0$; however, when a seismometer is operated so that $\rho_x \ll \rho \ll \infty$ measurements made as suggested by these conclusions should adequately determine all significant parameters of the seismometer. The equipment required would include a d-c ohmmeter, a signal generator, an oscilloscope, and an impedance bridge operating at a frequency well above f_s.

3.2.2 Transient response

So far, this chapter has considered only the response of the seismometer to sinusoidal excitation, i.e., particular solutions of Eq. 3.2.5. In the general solution additional terms, containing arbitrary constants, represent the transient response of the instrument to changes in the excitation. With step function excitation, as by sudden removal of a small weight from the bob, the transient terms represent the whole solution.

The transient response (or complementary function) is the sum of terms each of which is a solution of the differential Eq. 3.2.5 with the forcing function (left hand side) set to zero, i.e.,

$$0 = [D^3 + \rho D^2 + (\gamma^2 + 1)\ D + \rho]\ I \sqrt{L/S} \qquad (3.2.10)$$

It can be verified by repeated differentiation and substitution in Eq. 3.2.10 that a term of the form

$$Be^{-\beta\tau}, \quad \text{or} \quad Be^{-\beta\omega_o t}$$

is a solution, provided that β is a root of the algebraic equation:

$$0 = -\beta^3 + \rho\beta^2 - (\gamma^2 + 1)\ \beta + \rho\ . \qquad (3.2.10a)$$

The most general solution of a linear differential equation of the third order will contain three such exponential terms, with three independent constants B_1, B_2, and B_3 whose values are chosen to fit the initial or other boundary conditions.

Numerical evaluations of β, for various values of ρ and γ, to obtain solutions for Eq. 3.2.10 are readily performed using a nomogram, Fig. 3.1, constructed as follows:

Equation 3.2.10a may be rearranged to read

$$\rho = \beta + \frac{\gamma^2\beta}{\beta^2 + 1} \qquad (3.2.10b)$$

Let $$W = \frac{1}{\gamma^2}(\rho - \beta) \qquad (3.2.10c)$$

Then $$W = \frac{\beta}{\beta^2 + 1} \qquad (3.2.10d)$$

In Fig. 3.1.a W is plotted as ordinate against β as abscissa. Equation 3.2.10c gives a family of straight lines [such as (a), (b), (c), (d)] each member of which represents a unique pair of values for γ and ρ. For each member the intercept on the abscissa is ρ and the slope is $-1/\gamma^2$.

The curve on Fig.3.1.a is a plot of Eq. 3.2.10d which, it will be noted, is independent of both γ and ρ.

The intersections of the curve and each member of the family of straight lines give the real values of β which are solutions of Eq. 3.2.10a and therefore Eq. 3.2.10.

For any given instrument and coil configuration, γ is dependent almost entirely on magnet charge. The weaker the magnet the greater the slope of an appropriate straight line drawn in Fig.3.1.a while for increasing magnet strength the straight line approaches more nearly the horizontal.

For any given instrument and coil configuration, ρ is linearly related to the resistance external to the instrument: that is, as the external resistance is changed the straight line is translated parallel to itself. Increased external resistance represents a position further from the origin.

In order that the transient response of the instrument shall be aperiodic, i.e., contain no periodic term, it is necessary and sufficient that the three roots of Eq. 3.2.10 be real. Such a condition is shown on Fig.3.1.a by line (a). Note that for this value of γ there is a minimum value of ρ, or of the circuit resistance, below which there is but one intersection with the curve and therefore only one real root of Eq. 3.2.10a. The other two roots are complex and the transient response of the system contains a damped periodic term. This value of ρ corresponds to (b) of Fig.3.1.a. There is also a maximum value of ρ for this value of γ, above which the transient response contains a periodic term. This boundary corresponds to line (c) of Fig.3.1.a As the magnet charge of an instrument is lowered (γ is lowered) the range of ρ for which Eq. 3.2.10a has three real roots is decreased and there is a minimum value below which only one aperiodic solution for Eq. 3.2.10 can be found. The corresponding straight line is tangent to the curve at the inflection point (curve d of Fig. 3.1.a)

If β_1, β_2, and β_3, are the roots of Eq. 3.2.10a the operator on the right hand side of Eq. 3.2.10 may be rewritten in factored form as

$$(D + \beta_1)\ (D + \beta_2)\ (D + \beta_3) \tag{3.2.11a}$$

or

$$D^3 + (\beta_1 + \beta_2 + \beta_3) \cdot D^2 + (\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1)\ D + \beta_1\beta_2\beta_3 \tag{3.2.11b}$$

Comparison of coefficients between 3.2.11b and 3.2.10 shows that

$$\rho = \beta_1 + \beta_2 + \beta_3 = \beta_1\beta_2\beta_3 \tag{3.2.11c}$$

and

$$\gamma^2 + 1 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 \tag{3.2.11d}$$

It may be convenient (as for a check on calculations) to use the relation

$$\frac{f_s^2}{\rho} = \frac{\gamma^2 + 1}{\rho} = \frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} \tag{3.2.11e}$$

Using ρ and f_s as coordinates, Fig. 3.2 presents the field of variation of the parameters of the system, divided into regions on the basis of the character of the transient response. Combinations of parameters are marked to correspond to the lines in the nomogram (Fig. 3.1a) and to response curves in Figs. 3.3 through 3.9, the preparation of which is discussed in Section 3.2.3.

3.2.2.1 Solution for the weight-lift calibration

Since the arbitrary constants B_1, B_2 and B_3 of the transient terms depend entirely upon the boundary (usually initial) conditions, the general method will be illustrated by a detailed evaluation of the seismometer response to the step function frequently used for calibration, sometimes referred to as calibration by "weight-lift".

For this calibration, the initial conditions are:

$$\text{at} \quad t = 0, \; I = 0, \; \frac{dx}{dt} = 0, \; X = - mg;$$

$$t > 0, \; X = 0:$$

where m is the mass suddenly removed from the bob at $t = 0$. For this case we take $y = 0$ and $E = 0$, with ρ_x assumed $\ll \rho$. These boundary conditions substituted in Eq. 3.2.2 give the further condition that at $t \leq 0$, $dI/dt = 0$ and, when substituted in Eq. 3.2.1 for $t < 0$,

$$- mg = Sx \quad \text{or} \quad x = - \frac{mg}{S}$$

For $0 = y = E$ at all values of t, and $X = 0$ for $t > 0$ Eq. 3.2.1 becomes

$$0 = M \frac{d^2x}{dt^2} + Sx - GI$$

At $t = 0$ and by application of the boundary conditions,

$$\frac{d^2x}{dt^2} = \frac{mg}{M}$$

Differentiating 3.2.2 with respect to time under the above restrictions

$$0 = G \frac{d^2x}{dt^2} + L \frac{d^2I}{dt^2} + R \frac{dI}{dt}$$

and substituting the above boundary conditions it follows that,

$$\text{at} \quad t = 0, \ \frac{d^2I}{dt^2} = - \frac{Gmg}{LM}$$

Now, from the constants of the instrumental set-up, f_s and ρ may be evaluated. Using the nomogram, Fig. 3.1a, β_1 and possibly β_2 and β_3 are evaluated. Assuming that β_1, β_2, β_3, are real and distinct,

then $$I = B_1 e^{-\beta_1 \omega_o t} + B_2 e^{-\beta_2 \omega_o t} + B_3 e^{-\beta_3 \omega_o t} \qquad (3.2.12)$$

is a solution of Eq. 3.2.10. Knowing from the boundary conditions that at $t = 0$, $I = 0$ it follows that

$$0 = B_1 + B_2 + B_3$$

Taking the time derivative of Eq. 3.2.12 and knowing that at $t = 0$, $dI/dt = 0$ it follows that

$$0 = \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$$

The second time derivative of Eq. 3.2.12 is, by the boundary conditions, equal to $- mg\, G/ML$ from which

$$- mg\, G/ML\omega_o^2 = \beta_1^2 B_1 + \beta_2^2 B_2 + \beta_3^2 B_3$$

By algebraic manipulation of these three equations B_1, B_2 and B_3 may be expressed in terms of β_1, β_2 and β_3. Substitution in Eq. 3.2.12 gives

$$I = \frac{mgG}{\omega_o^2 ML} \frac{(\beta_2 - \beta_3)\, e^{-\beta_1 \omega_o t} + (\beta_3 - \beta_1)\, e^{-\beta_2 \omega_o t} + (\beta_1 - \beta_2)\, e^{-\beta_3 \omega_o t}}{(\beta_1 - \beta_2)(\beta_2 - \beta_3)(\beta_3 - \beta_1)} \tag{3.2.13}$$

which is the desired relation between current and time, for the weight-lift calibration.

It may happen that the calculated values of f_s and ρ are appropriate to a boundary of the aperiodic region; i.e., that two roots of the polynomial, say β_2 and β_3 are equal. In the nomogram (Fig. 3.1a) the straight line is tangent to the curve (lines b and c). The solution for the current is the limit of Eq. 3.2.13 as $\beta_2 \rightarrow \beta_3$. Equation 3.2.13 with $\beta_2 = \beta_3$ takes the indeterminate form 0/0.

The desired equation may be obtained either by the above process of evaluating B_1, B_2 and B_3 in a solution of the form

$$I = B_1 e^{-\beta_1 \omega_o t} + B_2 e^{-\beta_2 \omega_o t} + B_3 \omega_o t e^{-\beta_2 \omega_o t}$$

or by obtaining the limit of Eq. 3.2.13 as $\beta_2 \rightarrow \beta_3$. Either process leads to

$$I = \frac{mgG}{ML\omega_o^2} \left[\frac{e^{-\beta_2 \omega_o t} - e^{-\beta_1 \omega_o t}}{(\beta_1 - \beta_2)^2} + \frac{\omega_o t e^{-\beta_2 \omega_o t}}{(\beta_2 - \beta_1)} \right] \tag{3.2.14}$$

On the left (low resistance) side of the aperiodic region (Fig. 3.2), $\beta_1 < \beta_2$. On the right (high resistance) side of the aperiodic region $\beta_1 > \beta_2$.

At the downward pointing cusp of the boundary of the aperiodic region $(\rho = \sqrt{27}\ ,\ f_s = 3)$ the three roots are equal $(\beta_1 = \beta_2 = \beta_3 = \sqrt{3})$ and the solution reduces to:

$$I = \frac{mgG}{2ML}\, t^2 e^{-\sqrt{3}\,\omega_o t} \tag{3.2.15}$$

Again note that the condition for three equal roots can only be obtained for a particular value of γ (magnet charge and period) and a particular value of ρ (resistance, inductance, and period).

Outside the aperiodic region, two of the roots are complex conjugates. There will always be one real root of the cubic, obtainable from Fig. 3.1a. The complex roots may be written as $\sigma + j\nu$ and $\sigma - j\nu$, and evaluated from Eq. 3.2.11c, d and/or e, or estimated from Fig. 3.1b.

The solution may be written:

$$I = B_1 e^{-\beta_1 \omega_o t} + e^{-\sigma\omega_o t}\left(B_2 e^{j\nu\omega_o t} + B_3 e^{-j\nu\omega_o t}\right)$$

or

$$I = B_1 e^{-\beta_1 \omega_o t} + e^{-\sigma\omega_o t}\left(B_2' \cos \nu\omega_o t + B_3' \sin \nu\omega_o t\right)$$

Evaluation of B_1, B_2 and B_3 from the boundary conditions as before gives the equation for I as a function of t which may be written:

$$I = \frac{-\,mgG}{\omega_o^2 ML}\ \frac{\nu\left(e^{-\beta_1\omega_o t} - e^{-\sigma\omega_o t}\cos \nu\omega_o t\right) + (\beta_1 - \sigma)\, e^{-\sigma\omega_o t} \sin \nu\omega_o t}{\nu\,(\beta_1 - \sigma)^2 + \nu^3} \tag{3.2.16}$$

3.2.2.2 Transient response boundaries, loci and damping

The boundary between periodic and aperiodic response has been discussed earlier in Section 3.2.2.1 and is shown on Fig. 3.2.

The value $f_s = 1$ is also a boundary of the periodic region (corresponding to zero magnet charge). In principle the periodic region is also bounded by ρ = zero, although for a passive system the resistance of the coils sets a minimum value of ρ (about 1.5 in the Benioff variable reluctance instrument).

Another boundary of interest on Fig. 3.2 is the dotted line extending toward the origin from the cusp of the aperiodic region. This line is the locus of those values of f_s and ρ for which $\beta_1 = \sigma$, or $\rho = 3\beta_1$, and $\sigma 2 + \nu^2 = 3$. For this special case Eq. 3.2.16 may be written

$$I = \frac{-mgG}{SL} \nu^{-2} e^{-\beta_1 \omega_0 t} (1. - \cos \nu\omega_0 t)$$

This equation has the interesting property that for all positive values of t the current I is $\leq$ zero. The negative sign of the current arose in setting up the Lagrangian equations of motion when Eq. 2.4.4 for the magnetic energy was written with a positive sign for the cross-product term. The point at issue is that a plot of current vs time will show no zero crossings. It will be a damped sinusoid, tangent to zero for all values of $\nu\omega_0 t$ which are multiples of 2π, c.f. Fig. 3.3, curves 2. For values of f_s and ρ above and to the left of the dotted line in Fig. 3.2 there will be no zero crossings of the current resulting from a weight-lift test (c.f. Fig. 3.7, curves 10) while below and to the right, zero crossings will be observed.

Outside the aperiodic region the current resulting from a weight-lift test, as a function of time, may be looked upon as the sum of an exponentially decaying current and a damped oscillatory current. This may be shown by rewriting Eq. 3.2.16 as:

$$I = \frac{-mgG}{SL}\left\{\frac{\nu e^{-\beta_1\omega_o t} - e^{-\sigma\omega_o t}\left[\nu\cos\,\omega_o t - (\beta-\sigma)\,\sin\,\omega_o t\right]}{\nu(\beta_1 - \sigma)^2 + \nu^3}\right\} \qquad (3.2.16a)$$

which is of the form $I_{total} = I_{exp} + I_{osc}$

While the concept of damping as applied to Eq. 3.2.16 is somewhat obscure, it is feasible to consider the decrement of the oscillatory term as reflected in the excursions of the total current above and below its exponential component.

The current I_{osc} will be a maximum at some time t_1, it will again be a maximum at t_2 where $\nu\omega_o\ (t_2 - t_1) = 2\pi$.

The ratio of the contributions of the oscillatory term at times t_1 and t_2 is:

$$\frac{i_1}{i_2} = e^{\sigma\omega_o\ (t_2 - t_1)} = e^{2\pi\sigma/\nu}$$

and is the decrement of the oscillatory term, or

$$\ln\ \text{decrement} = 2\pi\sigma/\nu$$

The curve on Fig. 3.2 enclosing the aperiodic region is the locus of values of f_s and ρ for which $\sigma = \nu$; for the oscillatory term the logarithmic decrement = 2π, the decrement is 535 and the "overshoot" is $1/\sqrt{535}$ or 4.1% .

Obviously this line is only one of the family of loci of constant decrement. For decrements greater than 535 the curves hug the aperiodic region more closely than the curve shown, while for lesser decrements the curves are nearer the axes of the graph.

From this it follows that, having selected a mass, magnet charge, coil configuration and open circuit angular frequency (predetermined ω_o), adjustment of the load resistor to a predetermined decrement may occur at two values of external resistor (two values of ρ). It may well happen that the lower value of ρ is not physically realizable with a passive external resistor, since the resistance of the instrument coils sets a lower limit to the obtainable value of ρ.

If the magnet charge be reduced (as for instance to increase the "linear" range in the gap of a variable reluctance instrument) then a preselected decrement may not be obtainable by adjustment of the external resistor. As f_s decreases, the maximum obtainable decrement also decreases. The locus of this maximum is indicated on Fig. 3.2 as "maximum damping at constant f_s".

From Eq. 3.2.11c, remembering that $\beta'_2 = \sigma + j\nu$ and $\beta_3 = \sigma - j\nu$

$$\frac{4\sigma^2}{\sigma^2 + \nu^2} = \frac{(\rho - \beta_1)^2}{\rho/\beta_1}$$

Using Eq. 3.2.10b to eliminate ρ gives

$$\frac{4\sigma^2}{\sigma^2 + \nu^2} = \frac{\beta_1^2 \, (f_s^2 - 1)^2}{(\beta_1^2 + 1)\ (\beta_1^2 + f_s^2)}$$

When σ/ν is an extreme, $\dfrac{\sigma^2}{\sigma^2 + \nu^2}$ will also be an extreme. Since, at constant f_s, an extreme of σ/ν (maximum or minimum) as ρ varies is given by $\partial(\sigma/\nu)/\partial\rho = 0$ it follows that $\partial[\sigma^2/(\sigma^2 + \nu^2)]/\partial\rho = 0$ and since ρ is monotonically related to the single real root, β_1 of Eq. 3.2.10a in the periodic region it follows that $\partial[\sigma^2/\sigma^2 + \nu^2]/\partial(\beta_1^2) = 0$. Therefore, σ/ν is an extreme when $\beta_1^4 = f_s^2$ or $\rho = \beta_1^3$ and $\rho = f_s^{3/2}$ is the locus of maximum decrement in the region $f_s < 3$.

The value of the decrement on this locus is obtained from

$$(\gamma/v)_{max} = \left[\frac{4}{(f_s - 1)^2} - 1\right]^{-1/2}$$

3.2.3 Transient response curves, analog computer

In order to make more vivid the curves of current vs time after a weight-lift, when a seismometer is operated in the various regions of Fig. 3.2, an electronic analog computer was wired to solve Eqs. 3.2.1 and 3.2.2.

The curves, Figs. 3.3 through 3.9 show the results of weight lifts for the conditions given in Table 3.1. The numbered crosses on Fig. 3.2 show the appropriate location on this diagram for each group of curves.

Table 3.1

Constants for Figures 3.3 through 3.9
Benioff Variable Reluctance Seismometer

For all curves:

The upper coils are in parallel
The lower coils are in parallel
The two groups of coils are in series
The instrument resistance is 62.5 ohms
The circuit inductance is 6.8 henries
The mechanical damping, r_x, is assumed zero
The open circuit frequency is 1 cps
The mass of the bob is 107.5 kg
The full scale deflection, for the bob displacement is ~ 2.4 microns (one gm lifted from 107.5 kg), for bob velocity ~ 15 microns per second, for current ~ 30 microamperes

Approx. Field in gap	γ^2	f_s	ρ	Remarks see Fig. 3.2	External Resistance ohms	Fig. and curve
650	3	2	1.5		1.3	3.3 (1)
			2.12		27.6	(2)
			2.8	Max. damping	56	(3)
			6		192	(4)
1060	8	3	1.5		1.3	3.4 (5)
			4.28	$\sigma = \nu$	119	(6)
			5.2	Aperiodic	158	3.5 (7)
			6.3	$\sigma = \nu$	205	(8)
			10			3.6 (9)
1450	15	4	1.5		1.3	3.7 (10)
			5.67	$\sigma = \nu$	178	(11)
			8.0	Middle aperiodic	277	3.8 (12)
			11.48	$\sigma = \nu$	425	(13)
			20		787	3.9 (14)
			50		2060	(15)

3.2.4 Electrical analog of a seismometer

Equation 3.2.1 may be rewritten as

$$I = \frac{M}{G}\frac{d^2x}{dt^2} + \frac{r_x}{G}\frac{dx}{dt} + \frac{S}{G} - \frac{X}{G} + \frac{M}{G}\frac{d^2y}{dt^2}$$

or, taking $e = G\frac{dx}{dt}$

$$I = \frac{M}{G^2}\frac{d}{dt}\left(e + G\frac{dv}{dt}\right) + \frac{r_x}{G^2}e + \frac{S}{G^2}\int e\,dt - \frac{X}{G} \qquad (3.2.19)$$

It will be noted that the first term on the right side of Eq. 3.2.19 is mathematically identical to the current flowing through a capacitance, $C_x = M/G^2$, in series with a zero impedance generator of voltage, G dy/dt, when subjected to a voltage e. The second term may represent the current flowing through a resistance R_x, of magnitude G^2/r_x, when subjected to a voltage e. The third term is appropriate to the current flowing through an inductance, $L_x = G^2/S$, as a result of the voltage e. The fourth term may be considered as representing the current output, i, of an infinite impedance (constant current) generator, i = X/G.

Therefore, Eq. 3.2.19 is a valid expression for the currents in the following network:

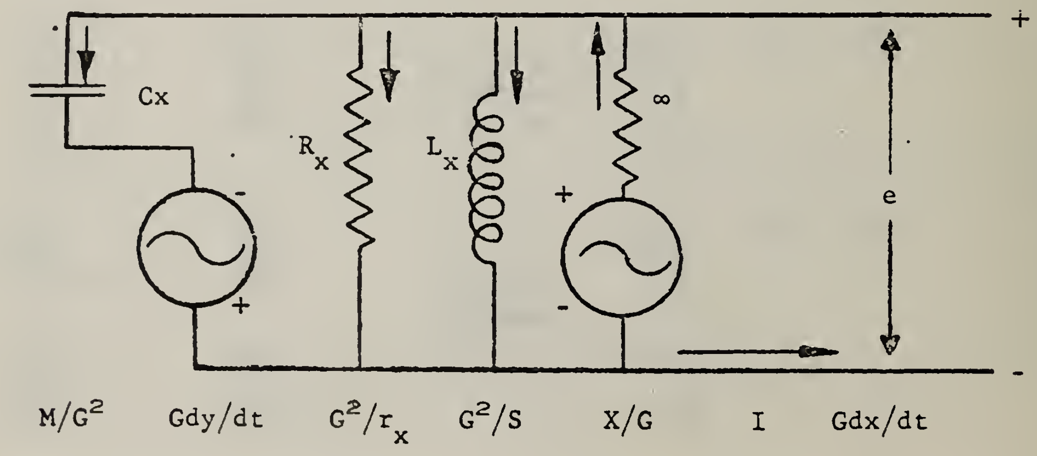


Fig. 3.10

and that, insofar as the mechanical portions of the seismometer influence the current flow in its electrical circuit, the mechanical portions may be replaced by the above circuit identically, with the directions of current flow and generator potentials taken positive as indicated.

As defined above, e is the open circuit voltage at the terminals of the seismometer so that, if the whole instrument be replaced by the following equivalent circuit Eq. 3.2.2 will apply in addition to Eq. 3.2.1.

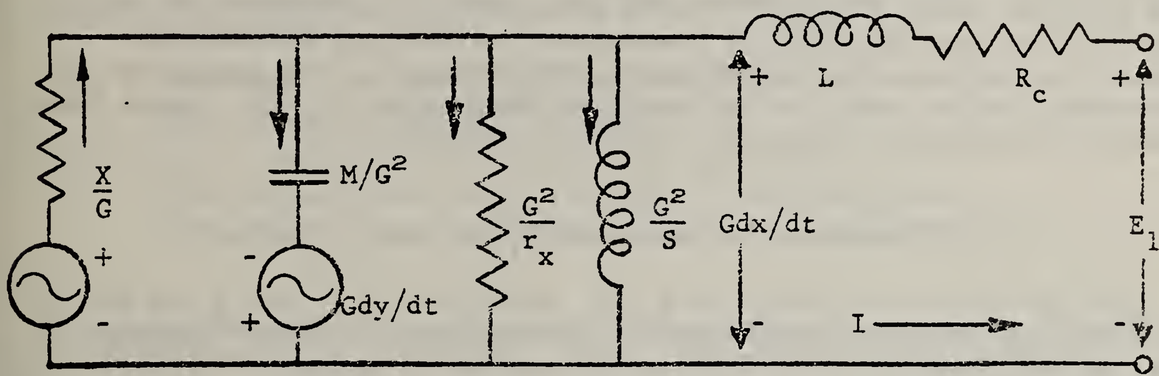


Fig. 3.11

Here E_1 is the potential difference at the terminals of the seismometer.

Section 4

D'ARSONVAL GALVANOMETER CONSIDERATIONS

4.1 General

With the possible exception of several methods for measuring the critical damping resistance of a galvanometer (subsection 4.4) the material of this chapter is better developed in such standard works as "Electrical Measurements" by Forest K. Harris. It is included here to show that the usual treatment of the galvanometer represents the special case of zero inductance in the development of Sections 2 and 3, to develop the galvanometer nomenclature used throughout the remainder of this report and to make the mathematical development of this report reasonably independent of specific literature references.

4.2 D'Arsonval Galvanometer, Equations of Motion

The derivations of Sections 2 (cf. subsection 2.9) and 3 are as valid for the d'Arsonval galvanometer as they are for electromagnetic inertial seismometers, provided the obvious substitutions of angular coordinates be made for the corresponding linear coordinates.

The basic equations corresponding to Eq. 3.2.1 and 3.2.2 are, therefore:

$$\Theta - K \frac{d^2\eta}{dt^2} = K \frac{d^2\theta}{dt^2} + r_\theta \frac{d\theta}{dt} + U\theta - G_3 I_3 \qquad (4.2.1)$$

and

$$E_3 = G_3 \frac{d\theta}{dt} + L_3 \frac{dI_3}{dt} + R_3 I_3 \qquad (4.2.2)$$

where

Θ = mechanical torque applied directly to the galvanometer coil,

θ = angular position of the coil relative to the case,

η = angular position of the galvanometer case, with respect to a "fixed" direction in space,

K = moment of inertia of the coil,

r_θ = mechanical (air) damping "constant",

U = stiffness of the coil suspension,

G_3 = galvanometer motor/generator constant,

I_3 = current in the galvanometer coil,

E_3 = the potential difference at the terminals of the galvanometer circuit,

L_3 = the electrical inductance of the galvanometer coil plus series inductance (if any),

R_3 = the electrical resistance of the galvanometer coil plus series resistance (if any), in any real case this will include the resistance of the suspensions.

As a galvanometer is normally used, the equations of motion, insofar as they relate the terminal voltage and/or current to angular position of the case or to mechanical torque applied to the coil, are generally of little interest. In special cases, a mechanical torque may be applied to the coil either deliberately or inadvertently, and the effect of Θ must be considered. Some sources of torque will be discussed in Section 8, Limitations in Principle and Practice.

The motor/generator constant, G_3, should not be confused with the "current sensitivity" usually given in catalogues. The latter is, by convention, related to U/G_3.

Equations 4.2.1 and 4.2.2 may be combined to eliminate I_3. With θ and L_3 presumed zero, η constant, and defining $\mathcal{R}_g$, λ_g and ω_g as

$$\mathcal{R}_g \doteq G_3^2/2\omega_g K, \quad \lambda_g \equiv G_3^2/2R_3\omega_g K \quad \text{and} \quad \omega_g = \sqrt{U/K}$$

the result may be written

$$\sqrt{2\lambda_g\omega_g}\,E_3/\sqrt{R_3} = \left[\frac{d^2}{dt^2} + \left(\frac{r_\theta}{\omega_g K} + 2\lambda_g\right)\omega_g\frac{d}{dt} + \omega_g^2\right]\theta\sqrt{K} \qquad (4.2.3)$$

It should be pointed out that, given an ideal instrument, i.e., one with no mechanical losses, $\mathcal{R}_g$ is the critical damping resistance (CDRX plus coil resistance) which one would measure by determining, for instance, the maximum circuit resistance adequate to suppression of overshoot of the coil when excited by square waves at a frequency which is low compared to ω_g. Such measurements on a real instrument will lead to a higher value of circuit resistance than $\mathcal{R}_g$ because the coil motion will be critically damped when $r_\theta/2\omega_g K + \mathcal{R}_g/R_3 = 1$, that is, when λ_g plus the fraction of critical damping represented by mechanical losses equals unity.

4.3 Galvanometer With a Resistive Load

4.3.1 Transient response

In Fig. 3.2 the region of greatest mathematical interest is that lying around the cusp of the curve bounding the aperiodic region. This is also the region in which the variable reluctance pick-up seismometer is generally operated, as a compromise between linearity in the gap, generator constant, generator resistance and generator inductance. In the d'Arsonval galvanometer somewhat different compromises result in a generator/motor with a much lower inductance and, often a relatively higher internal resistance[1]. That is, the value of f_s for the galvanometer is, in general, much higher than for the seismometer; as is ρ, for a given value of circuit resistance; both reflecting the lowered value of coil inductance. Consider the ratio

$$\frac{f_s^2 - 1}{2\rho} = \frac{G_3^2}{2LU} \cdot \frac{L\omega_g}{R} = \frac{G_3^2}{2K\omega_g R} = \frac{\mathcal{R}_g}{R}$$

This quantity $\mathcal{R}_g/R$ is the ratio of the electrodynamic damping to critical. (This appears as the quantity $\gamma - \gamma_o$ in the discussion of the galvanometer by Harris, cf p.59.)

In Section 3, Fig. 3.8 and 3.9, the curves of groups 13, 14, and 15 are seen to resemble the response of a galvanometer for $\lambda_g = 0.65$, 0.375, and 0.15 respectively. The most conspicuous difference is the rounding of the initial break. For larger values of f_s the resemblance would be even closer. For instance, while a reluctance pick-up may be operated with f_s approaching 3 and with ρ as low as 1.5, a high sensitivity d'Arsonval galvanometer will have $f_s \gtrsim 600$ and a minimum obtainable value of $\rho \gtrsim 10^5$. These values are well beyond the upper right corner of Fig. 3.2, where the difference between the third order differential equation and the second order is very slight.

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[1] These remarks are not applicable to the Thompson moving magnet galvanometer.

4.3.2 Response to sinusoidal electrical excitation

Throughout the remainder of this section we shall be concerned chiefly with the motions of the galvanometer mirror in response to sinussoidal electrical excitation and with the influence of the galvanometer on the electrical circuits to which it is connected.

In line with the discussion of Sections 4.2 and 4.3.1 we will consider η to be constant and Θ, r_θ, and L_3 to be zero. Equations 4.2.1 and 4.2.2 now take the form

$$0 = K \frac{d^2\theta}{dt^2} + U\theta - G_3 I_3 \tag{4.3.1}$$

and

$$E_3 = G_3 \frac{d\theta}{dt} + R_3 I_3 \tag{4.3.2}$$

Considered as a two terminal electrical network the current-voltage relation for a galvanometer may be obtained either from Eqs. 4.3.1 and 4.3.2 above or from Eq. 3.2.9 by letting L → 0. Either approach yields

$$\frac{E_3}{I_3} = R_3 + j\omega \frac{G_3^2}{(\omega_g^2 - \omega^2)\, K} \tag{4.3.3}$$

Since the critical damping resistance, $\mathcal{R}_g$, is given by $\mathcal{R}_g = G_3^2/2\sqrt{UK}$ (cf. paragraph 4.3.1) Eq. 4.3.3 may be written

$$\frac{E_3}{I_3} = R_3 + 2j\mathcal{R}_g \frac{1}{\frac{\omega_g}{\omega} - \frac{\omega}{\omega_g}} \qquad (4.3.4)$$

Equation 4.3.4 has been derived assuming that mechanical losses may be neglected. While this assumption will be used in developing the response of an overall system consisting of a seismometer coupled by a resistive network to a galvanometer, it may not always be realized in practice. In particular, in measuring the parameters of a galvanometer a more exact relation may be useful.

By analogy with the equivalent circuit of a seismometer (paragraph 3.2.4) one may sketch an equivalent circuit and assign nomenclature for a galvanometer, ignoring the generators representing case motion and mechanical forces applied directly to the coil as well as electrical inductance associated with the coil. The result is Fig. 4.1.

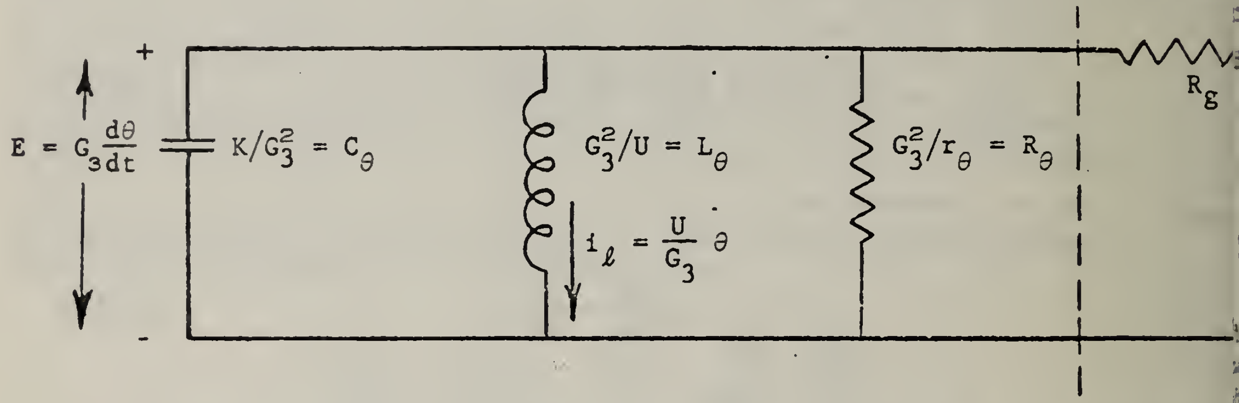

+
$E = G_3 \frac{d\theta}{dt}$
-
$K/G_3^2 = C_\theta$
$G_3^2/U = L_\theta$
$i_\ell = \frac{U}{G_3}\theta$
$G_3^2/r_\theta = R_\theta$
R_g

Figure 4.1

The quantity $\mathcal{R}_g$ which we have been calling the critical damping resistance turns out to be half the quantity $\sqrt{L_\theta/C_\theta}$, a familiar parameter in the theory of tuned circuits.

The impedance, Z_θ, of the circuit to the left of the dotted line is given by

$$\frac{1}{Z_\theta} = \frac{1}{R_\theta} + \frac{1}{j\omega L_\theta} + j\omega C_\theta$$

which by algebraic manipulation, remembering that

$$L_\theta C_\theta = (G_3^2/U)\ (K/G_3^2) = 1/\omega_g^2 \quad \text{and that} \quad L_\theta$$

may be written $2\mathcal{R}_g/\omega_g$, yields

$$Z_g = R_g + R_\theta \frac{1 + j\,\dfrac{R_\theta}{\mathcal{R}_g}\left(\dfrac{\omega_g^2 - \omega^2}{2\omega_g\omega}\right)}{1 + \dfrac{R_\theta^2}{\mathcal{R}_g^2}\left(\dfrac{\omega_g^2 - \omega^2}{2\omega_g\omega}\right)^2} \qquad (4.3.5)$$

for the impedance of a galvanometer considered as a two terminal electrical network. It should be noted that at $\omega = \omega_g$ the reactive term drops out, giving

$$Z_g \Big|_{\omega = \omega_g} = R_g + R_\theta \qquad (4.3.6)$$

It will be convenient in Section 4.4 to have available expressions for the impedance of series and parallel combinations of a galvanometer with a capacitor or an inductor. The remainder of this section is devoted to developing these relations.

If a capacitor be placed in series with a galvanometer the impedance of the combination is given by

$$Z_{sc} = R_g + \frac{2\mathcal{R}_g \omega C_e \frac{\omega^2}{\omega_g^2} - j\left[\frac{R_\theta}{2\mathcal{R}_g}\left(1 - \frac{\omega^2}{\omega_g^2}\right)^2 + \frac{2\mathcal{R}_g}{R_\theta}\frac{\omega^2}{\omega_g^2} - R_\theta \omega C_e \frac{\omega}{\omega_g}\left(1 - \frac{\omega^2}{\omega_g^2}\right)\right]}{\omega C_e\left[\frac{R_\theta}{2\mathcal{R}_g}\left(1 - \frac{\omega^2}{\omega_g^2}\right)^2 + \frac{2\mathcal{R}_g}{R_\theta}\frac{\omega^2}{\omega_g^2}\right]} \quad (4.3.7)$$

where Z_{sc} is the impedance of the combination and C_e is the electrical capacitance of the added capacitor. There is an angular frequency, ω_{sc}, at which the imaginary part of this equation is zero. Again algebraic manipulation produces the relation

$$\omega_{sc} C_e \mathcal{R}_g = \frac{\omega_g^2 - \omega_{sc}^2}{2\omega_g \omega_{sc}} + \frac{\mathcal{R}_g^2}{R_\theta^2}\,\frac{2\omega_g \omega_{sc}}{(\omega_g^2 - \omega_{sc}^2)} \quad (4.3.8)$$

which, for $R_\theta > \mathcal{R}_g$ may be solved for $\mathcal{R}_g$ by successive approximation.

If a capacitor is placed in parallel with a galvanometer, there is an angular frequency, ω_{pc}, at which the imaginary part of the impedance of the combination is zero. For this frequency the following relation holds:

$$\mathcal{R}_g = \frac{\omega_g^2 - \omega_{pc}^2}{4\omega_g \omega_{pc}^2 C_e}\left(\frac{R_\theta}{R_g + R_\theta}\right)^2 \left[1 + \sqrt{1 - 4\omega_{pc}^2 C_e^2 R_g^2 \left(\frac{R_g + R_\theta}{R_\theta}\right)^2}\,\right] \quad (4.3.9)$$

Note that ω_{pc} is real provided that

$$\frac{R_\theta}{R_g + R_\theta} \geq 2\omega_{pc} C_e R_g \quad (4.3.9a)$$

The combination of an inductor in series with a galvanometer has already been considered in paragraph 3.2.1.3 for the special case of $R_\theta \rightarrow \infty$, $(r_x \rightarrow 0)$. The reactive term vanishes at ω_s from which

$$\mathcal{R}_g = \frac{L}{2\omega_g}(\omega_s^2 - \omega_g^2) \tag{4.3.10}$$

Inclusion of the mechanical losses as an equivalent shunt resistor, R_θ, leads to

$$\mathcal{R}_g = \frac{\omega_s^2 - \omega_g^2}{4\omega_g\omega_s^2 L} R_\theta^2 \left[1 - \sqrt{1 - \frac{4\omega_s^2 L^2}{R_\theta^2}}\right] \tag{4.3.10a}$$

Note that neither the coil resistance nor the resistance of the added inductor appears in this equation. For R_θ large with respect to $\omega_s L$ series expansion of the radical leads to Eq. 4.3.10 so that, within engineering accuracy, for $R_\theta \geq 10\omega_s L$ Eq. 4.3.10 is to be preferred. It should also be noted that for ω_s to be real it is necessary that

$$R_\theta > 2\omega_s L \tag{4.3.10b}$$

Consider a physical inductor having resistance, R_L, in parallel with a galvanometer. The combination has an impedance whose reactive component vanishes at an angular frequency, ω_{PL}, for which

$$\mathcal{R}_g = \frac{\omega_{PL}^2 - \omega_g^2}{2\omega_{PL}\omega_g} \left[\frac{R_\theta}{R_g + R_\theta}\right]^2 \frac{R_L^2 + \omega_{PL}^2 L^2}{2\omega_{PL} L} \left[1 + \sqrt{1 - \left(\frac{R_g + R_\theta}{R_\theta}\right)^2 \left(\frac{2\omega_{PL} L}{R_L^2 + \omega_{PL}^2 L^2}\right)^2 R_g^2}\right] \tag{4.3.11}$$

For real values of ω_{PL} it is necessary that

$$\frac{R_L^2 + \omega_{PL}^2 L^2}{2\omega_{PL} L} \geq \frac{R_g + R_\theta}{R_\theta} R_g \qquad (4.3.11a)$$

4.4 Determination of Galvanometer Parameters

As the galvanometer is commonly used, the angular position of the mirror is observed with a lamp and scale or a telescope and scale. The determination of the operating parameters of such a galvanometer by combined optical and electrical measurements is not difficult in principle. In some cases one cannot observe the angular deflection of the galvanometer mirror (as for instance when combined with a photoelectric amplifier) and the galvanometer parameters must be inferred from measurements on the input and output electrical terminals without direct measurement of the angular position of the coil.

In subsequent sections it will be shown that, in the equations for the overall frequency response of seismometer-galvanometer combinations, the galvanometer characteristics appear in various combinations, all of which can be expressed in terms of ω_g, $\mathcal{R}_g$, and K. The determination of ω_g and $\mathcal{R}_g$ is discussed in Sections 4.4.1 and 4.4.2. In an assembled galvanometer without optical or mechanical access to the coil, a procedure resulting in the evaluation of K, the moment of inertia of the suspended system, has been sought unsuccessfully. This problem is resolved in the following paragraph. The measurement of a single mechanical quantity, such as a torque or deflection, would provide the necessary relation between electrical and mechanical parameters. For instance, when the coil can be observed, Eq. 4.2.3 at zero frequency becomes

$$\sqrt{2\mathcal{R}_g \omega_g}\, \frac{E_3}{R_3} = \omega_g^2 \sqrt{K}\, \theta \doteq \omega_g \sqrt{U}\, \theta \qquad (4.4.1)$$

so that if $\mathcal{R}_g$ and ω_g be measured by any means, a single determination of the coil deflection for a known d-c current serves to evaluate the moment of inertia, K, and the suspension stiffness, U.

4.4.1 Galvanometer-photoelectric amplifier combination

For one reason or another one may prefer not to record the deflections of the galvanometer directly but rather to convert them to an electrical signal. If one presumes that the output of such a device is linear with mirror rotation, and is not frequency dependent, then it may be considered to have the transfer relation $E_o = k\theta$. Here k is a constant having the units volts/radian, cps/radian, etc. as may be appropriate.

In the overall system response curves of Sections 5 and 6 and to some extent Section 7 the output has been expressed as θ which always appears in a term of the form $\sqrt{K}\,\theta$. When a system, otherwise appropriate to one of these equations, is used with a photo-electric amplifier then $\sqrt{K}\,\theta$ must be replaced by its equal $E_o\sqrt{K}/k$.

It is easily shown that the equation corresponding to 4.4.1 which is appropriate to a galvanometer-photoelectric amplifier combination is, at zero frequency,

$$\sqrt{2\mathcal{R}_g\omega_g}\,\frac{E_3}{R_3} = \omega_g^2\,\frac{\sqrt{K}}{k}\,E_o \qquad (4.4.2)$$

Therefore as far as the determination of galvanometer moment of inertia is concerned we find the happy circumstance that, when one is recording galvanometer deflections directly, the galvanometer is accessible and Eq. 4.4.1 may be used. When the galvanometer is not accessible one is recording some other variable related to its deflection and one needs to know $\sqrt{K}/k$ which may be evaluated using Eq. 4.4.2.

Given a galvanometer combined with a photoelectric amplifier whose output voltage follows the relation $E_o = k\theta$, where k is independent of frequency, one may proceed as follows:

First: Apply a momentary voltage to the terminals and record the output voltage as a function of time with the input circuit open.

From the record obtained, compute the logarithmic decrement. Depending upon the intended use of the instrument, this may or may not justify the suppression of r_θ, as has been done in most of this report.

Determine the time required for a convenient number of cycles of oscillation of the galvanometer. From this, ω_g is readily computed.

Second: Establish a known direct current, i_3, in the galvanometer input circuit and measure the resulting voltage, e_o, at the output. If s be the galvanometer sensitivity in amp/radian (U/G_3) then:

$$i_3 = s\theta = Ue_o/G_3k \quad \text{or} \quad \frac{e_o}{i_3}\Big|_{dc} = \frac{G_3k}{U} \tag{4.4.3a}$$

Third: Energize the galvanometer with a sinusoidal voltage e_3 at ω_g. The value of ω_g computed above may be used or the voltage source may be adjusted to that frequency for which i_3, at constant e_3, is a minimum. Measure the output voltage e_o then:

$$e_3 = G_3\dot{\theta} = |\,G_3\omega_g\theta\,| = |\,G_3\omega_g e_o/k\,| \quad \text{or} \quad \Big|\frac{e_3}{e_o}\Big|_{\omega_g} = \frac{\omega_g G_3}{k} \tag{4.4.3b}$$

Combining Eq. 4.4.3a and 4.4.3b gives:

$$\frac{e_3}{e_o}\Big|_{\omega_g}\frac{e_o}{i_3}\Big|_{dc} = \omega_g G_3^2/U = 2\mathcal{R}_g \tag{4.4.4}$$

4.4.2 Galvanometer measured as a two terminal network

An alternative procedure for measuring $\mathcal{R}_g$ and ω_g involves the use of a high gain cathode ray oscilloscope and quality capacitor or inductor rather than the voltmeter required above. It is more general in that only the galvanometer input terminals need be accessible.

If the galvanometer under test, a signal generator and a suitable oscilloscope are connected as shown in Fig. 4.2 with the switch in the Y position then at one and only one frequency will a straight line pattern appear on the CRT face.

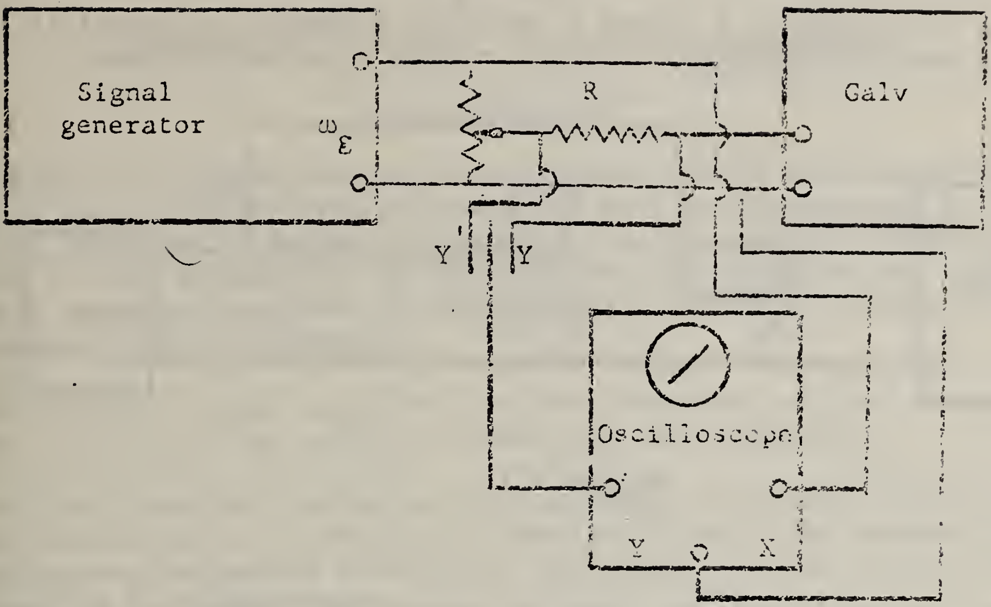


Figure 4.2

From Eqs. 4.3.5 and 4.3.6 it can be seen that this frequency is that for which $\omega = \omega_g$ and I_3 is a minimum. Let the tilt of this closed loop be expressed in mm or mv as convenient. The switch is then moved to the Y' position and the tilt is again measured. Then

$$\frac{Y}{Y' - Y} = \frac{R_e + R_g}{R} \quad \text{or} \quad R_e + R_g = RY/(Y' - Y) \qquad (4.4.5)$$

where Y' and Y are the numerical values of the tilts measured in the corresponding switch positions R_g the d-c resistance of the galvanometer, may be measured by any convenient method.

Addition of a capacitor or inductor to the galvanometer and a second measurement using the circuit of Fig. 4.2 with the switch in the Y position gives all the information necessary to compute [illegible], the critical damping resistance, using whichever of Eqs. 4.3.8 through 4.3.11 is appropriate.

Alternatively we have also used the bridge circuit sketched in Fig. 4.3 to determine both ω_g and $R_\theta + R_g$. We have used an oscilloscope with a differential Y input as detector although, in principle, any detector with a "floating", or differential, input may be used.

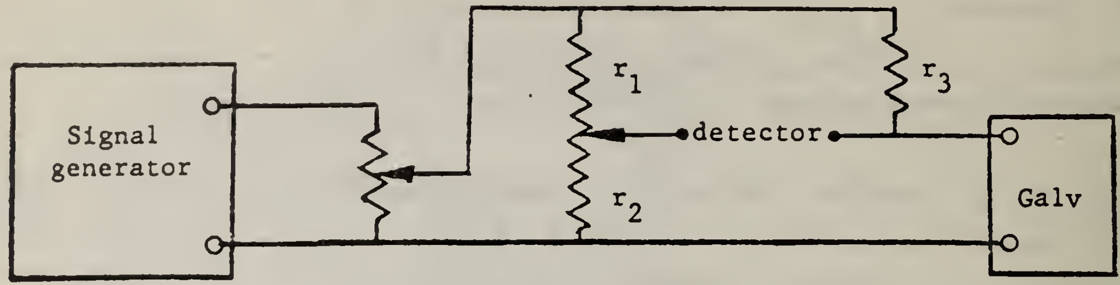


Figure 4.3

The detector will indicate a null or balance only when $\omega = \omega_g$ and $r_1/r_2 = r_3/(R_\theta + R_g)$.

If the galvanometer sketched above is replaced by a galvanometer and reactor (vide supra) then balance is obtained only at the frequency appropriate to a determination of $\mathcal{R}_g$. The reader should be cautioned that in this case $r_2 r_3/r_1$ will not give the correct value for $(R_\theta + R_g)$ to use elsewhere in this report.

Section 5

SEISMOMETER, RESISTIVE T PAD AND GALVANOMETER

5.1 General

In this section equations are developed for the performance of a variable reluctance seismometer connected to a galvanometer by a resistive T pad. The equations are based on the assumption that mechanical damping of both seismometer and galvanometer may be neglected. Also implicit in these equations is the assumption that the inductance of the galvanometer may be neglected.

Attempts to obtain equations for the transient solution analogous to Eqs. 3.2.13, 3.2.14, and 3.2.15 have not been successful.

We shall discuss limitations on the design of an adjustable T pad without the formality of explicitly displaying amplitude-frequency or phase-frequency equations since their development by the methods used in Section 3 is straightforward.

5.2 Seismometer with Bilateral Coupling to the Galvanometer

A commonly used arrangement consists of a seismometer, a three terminal network, and a galvanometer. The three terminal network is usually resistive, arranged to provide adjustable current attenuation; and some attempt is usually made to present a constant load to both the seismometer and galvanometer.

Since any three terminal resistive network may be replaced by an equivalent T network, we may sketch such a network (Fig. 5.1) and write the equations of motion.

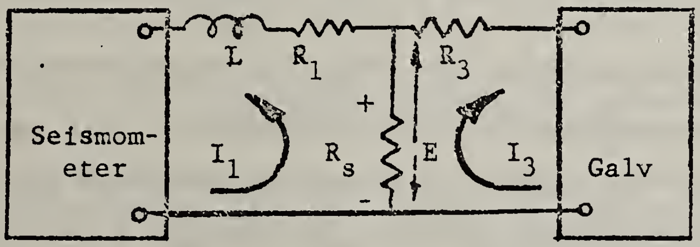


Figure 5.1

They are:

$$X - M\frac{d^2y}{dt^2} = M\frac{d^2x}{dt^2} + Sx - G_1I_1 \qquad (5.2.1)$$

$$E = G_1\frac{dx}{dt} + L\frac{dI_1}{dt} + R_1I_1 \qquad (5.2.2)$$

$$0 = K\frac{d^2\theta}{dt^2} + U\theta - G_3I_3 \qquad (5.2.3)$$

$$E = G_3\frac{d\theta}{dt} + R_3I_3 \qquad (5.2.4)$$

$$E = -(I_1 + I_3)R_s \qquad (5.2.5)$$

R_1 is the sum of the resistance in the input arm of the T and the resistance of the seismometer coils, R_c. It therefore has a minimum value set by the resistance of the coils themselves.

R_3 is the sum of the resistance in the output arm of the T and the resistance of the galvanometer coil. It therefore has a minimum value set by the galvanometer coil.

R_s is the shunt resistor of the T pad and may have any positive value.

The other symbols have the same significance as previously assigned them.

Eliminating E, I_1, I_3 and x from these five equations gives

$$
\begin{aligned}
&\frac{R_s}{R_1R_s + R_1R_3 + R_3R_s}\sqrt{\frac{G_1^2G_3^2}{MK}}\,\frac{d}{dt}\left(\frac{X}{\sqrt{MK}} - \sqrt{\frac{M}{K}}\,\frac{d^2y}{dt^2}\right) = \\
&\left\{\frac{R_3 + R_s}{R_1R_s + R_1R_3 + R_3R_s}\, L\, \frac{d}{dt}\left[\frac{d^4}{dt^4} + \left(\omega_o^2 + \frac{G_1^2}{ML} + \omega_g^2\right)\frac{d^2}{dt^2} + \left(\omega_o^2 + \frac{G_1^2}{ML}\right)\omega_g^2\right] + \right. \\
&\frac{R_1 + R_s}{R_1R_s + R_1R_3 + R_3R_s}\cdot\frac{G_3^2}{K}\cdot\frac{d}{dt}\left[\frac{d^2}{dt^2} + \omega_o^2\right] + \qquad (5.2.6) \\
&\frac{1}{R_1R_s + R_1R_3 + R_3R_s}\, L\, \frac{G_3^2}{K}\,\frac{d^2}{dt^2}\left[\frac{d^2}{dt^2} + \omega_o^2 + \frac{G_1^2}{ML}\right] + \\
&\left.\left[\frac{d^4}{dt^4} + (\omega_o^2 + \omega_g^2)\frac{d^2}{dt^2} + \omega_o^2\omega_g^2\right]\right\}\theta
\end{aligned}
$$

A "mean coupling coefficient" may be defined as follows:

With the galvanometer coil blocked, the fraction of the current generated by the seismometer which passes through the galvanometer coil is given by $R_s/(R_3 + R_s)$. Likewise if the seismometer mass be blocked and its inductance neglected then the fraction of the current generated by motions of the galvanometer coil which passes through the seismometer is given by $R_s/(R_1 + R_s)$. Each of these may be looked upon as a coupling coefficient whose value lies between zero and one. We may therefore define a mean coupling coefficient

$$\alpha \equiv R_s/\sqrt{(R_1 + R_s)\ (R_3 + R_s)}, \quad 0 \leq \alpha \leq 1$$

This definition of α has been based on current attenuation in the forward and reverse directions with the appropriate mass blocked. One may equally well define the same mean coupling coefficient in terms of the forward and reverse voltage attenuations with the appropriate output open. Alternatively suppose that a d-c e.m.f. is introduced in either the seismometer or galvanometer branch of the circuit. The current in the other branch dissipates a power, W_3, in the network. If the total power put into the network is W_1, then the ratio W_3/W_1 is α^2 according to either of the above definitions.

From the concept of the short circuit frequency developed in paragraph 3.2.1.1 we may, as before, define

$$\omega_s \equiv \omega_o \sqrt{1 + G_1^2/LS} \quad \text{and} \quad f_s \equiv \omega_s/\omega_o$$

The defining equation for ρ given in paragraph 3.2 here takes the form

$$\rho \equiv \left[R_1 + \frac{R_3 R_s}{R_3 + R_s}\right] \frac{1}{\omega_o L} = \frac{R_1 R_3 + R_1 R_s + R_3 R_s}{(R_3 + R_s)\ \omega_o L}$$

where the bracket is the resistance of the circuit as seen by the seismometer when the galvanometer armature is blocked.

The "ratio to critical damping", i.e., $\lambda_g = \mathcal{R}_g/R$, as commonly used in analyzing circuits involving galvanometers will be used here with the defining equation

$$\lambda_g = \frac{G_3^2}{2\sqrt{UK}\left[R_3 + \dfrac{R_1 R_s}{R_1 + R_s}\right]} = \frac{G_3^2 \quad (R_1 + R_s)}{2\omega_g K\ (R_1 R_3 + R_1 R_s + R_3 R_s)}$$

The reader will note that this definition is equivalent to defining λ_g as the fraction of critical damping seen by the galvanometer if the seismometer is replaced by a non-inductive resistor of value equal to the coil resistance.

Since $\omega_g = \sqrt{U/K}$ as before and using $D = d/\omega_o dt$, Eq. 5.2.6 may be manipulated to the alternative forms 5.2.7 and 5.2.8 with $f_g = \omega_g/\omega_o$

$$\frac{\alpha\sqrt{\rho 2\lambda_g f_g}}{\omega_o}\,\gamma D\left(\frac{X}{\sqrt{S}} - \sqrt{M}\,D\dot{y}\right)$$

$$= \left[(D^3 + \rho D^2 + f_s^2 D + \rho)\ (D^2 + 2\lambda_g f_g D + f_g^2) - \alpha^2 2\lambda_g f_g D^2\ (D^2 + f_s^2)\right]\ \theta\sqrt{K} \tag{5.2.7}$$

$$\left\{(D^3 + \rho D^2 + f_s^2 D + \rho)\ [D^2 + 2\lambda_g f_g\ (1 - \alpha^2)\ D + f_g^2] + \alpha^2 2\lambda_g f_g \rho D\ (D^2 + 1)\right\}\ \theta\sqrt{K} \tag{5.2.8}$$

5.2.1 Influence of variable attenuation on the over-all response-frequency characteristics

The over-all gain of a seismometer-galvanometer combination is usually adjusted by altering the resistive T pad. Unfortunately it is not possible to do so without some alteration of the shape of the response-frequency characteristics. This can be seen from Eqs. 5.2.7 and 5.2.8. In both cases, the coupling coefficient α appears as a factor on the left hand side, so that θ, for any given excitation is proportional to it. The right hand side of Eq. 5.2.7 contains two terms, the first of which does not contain α, while the second is proportional to α^2. The first term is the product of two factors, of which one appears in the equation for the response of a seismometer connected to a resistive load, (Eq. 3.2.5) and the other in the equations for the galvanometer (Eq. 4.2.3). If the T pad maintains constant input and output resistance, then ρ and λ_g are constant. The term in α^2 will affect the second and fourth derivatives only.

Alternatively a resistive network may be designed to have ρ and $R_3 + R_s$ constant. Here $(1 - \alpha^2)\ \lambda_g$ is constant and Eq. 5.2.8 shows that α^2 affects the first and third derivatives. In neither case are the constant term and the fifth derivative affected.

In the first case, constant λ_g, the frequencies at which the earth velocity is in phase, or 180° out of phase, with the output are independent of α, or the attenuator setting. The relative gain at these frequencies is dependent on α. Conversely the relative gain at the frequencies of ± 90° phase shift are independent of α but the frequency at which this phase relation occurs is α dependent.

In the second case, with $R_3 + R_s$ constant the frequencies at which the earth displacement is in phase, or 180° out of phase with the output are functions of the T pad setting while the relative gain at these frequencies is not. Conversely, the frequencies at which the output is 90° out of phase with the earth velocity are independent of the attenuator setting, but the relative gain at these frequencies will depend upon attenuator setting.

As a practical problem one may set tolerance limits on the permissible variations of the response-frequency curves and, for a specific seismometer and galvanometer, compute the maximum value of α (and of system gain) attainable without exceeding the prescribed limits.

The greatest range of gain settings for a prescribed variation of the relative response-frequency curves will be obtained using a judicious compromise between these two ways of restricting the T pad parameters.

In any case, as $R_s \rightarrow 0$, $\alpha \rightarrow 0$ also and the shape of the response-frequency curve becomes that expected from the two electromechanical systems considered as filters, connected by some unilateral network such as an idealized cathode-follower. The amplitude of the overall system gain also approaches zero.

5.2.2 Direct coupling of seismometer and galvanometer

Since currents flowing in the shunt arm of a T pad represent an energy loss it becomes of interest to examine the special case of direct coupling for possible use as a high gain system of minimal size or weight. In this case the interaction of the seismometer and galvanometer is greatest since all the current in the transducer also passes through the galvanometer. The circuit may be sketched as shown in Fig. 5.2 wherein the inductance L may be distributed between the seismometer and galvanometer in any manner.

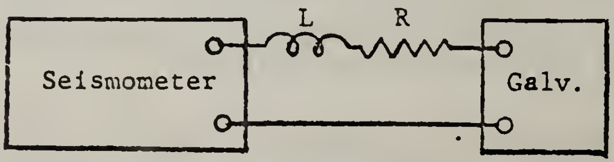


Figure 5.2

The resistor R, ($R_1 + R_3$ of Section 5.2) includes the coil resistance of both the seismometer and galvanometer as well as any resistance in the connecting cable.

As $R_s \to \infty$ in Eq. 5.2.6, the third of the four terms on the right hand side vanishes because it contains R_s in the denominator only. The equation can be rearranged to the following form, in which even and odd orders of the derivative are separated:

$$\sqrt{\frac{G_1^2}{M}\frac{G_3^2}{K}}\frac{d}{dt}\left(\frac{x}{\sqrt{MK}} - \sqrt{\frac{M}{K}}\frac{d^2y}{dt^2}\right) =$$

$$\left[L\frac{d}{dt}\left(\frac{d^2}{dt^2}+\omega_o^2\right)\left(\frac{d^2}{dt^2}+\omega_g^2\right) + \frac{G_1^2}{M}\frac{d}{dt}\left(\frac{d^2}{dt^2}+\omega_g^2\right) + \frac{G_3^2}{K}\frac{d}{dt}\left(\frac{d^2}{dt^2}+\omega_o^2\right)\right.$$

$$\left. + R\left(\frac{d^2}{dt^2}+\omega_o^2\right)\left(\frac{d^2}{dt^2}+\omega_g^2\right)\right]\theta \qquad (5.2.9)$$

For sinusoidal excitation, and with no forces applied directly to the mass, one may substitute $j\omega$ for d/dt in Eq. 5.2.9 to obtain:

$$\sqrt{\frac{G_1^2G_3^2}{MK}}\sqrt{\frac{M}{K}}\frac{y}{\theta} = L\frac{(\omega_o^2-\omega^2)(\omega_g^2-\omega^2)}{\omega^2} + \frac{(\omega_g^2-\omega^2)}{\omega^2}\frac{G_1^2}{M} + \frac{(\omega_o^2-\omega^2)}{\omega^2}\frac{G_3^2}{K}$$

$$- j\frac{R(\omega_o^2-\omega^2)(\omega_g^2-\omega^2)}{\omega^3} \qquad (5.2.10)$$

Since the imaginary part of Eq. 5.2.10 vanishes at $\omega = \omega_o$ and $\omega = \omega_g$ it follows that the galvanometer deflection is in phase (or 180° out of phase) with the earth displacement at $\omega = \omega_o$ and $\omega = \omega_g$.

The sensitivity at ω_o may be computed from

$$\frac{y}{\theta}\bigg|_{\omega=\omega_o} = \frac{G_1^2}{M}\sqrt{\frac{K}{M}}\sqrt{\frac{MK}{G_1^2G_3^2}}\left(\frac{\omega_g^2}{\omega_o^2} - 1\right) \qquad (5.2.11)$$

while at ω_g the sensitivity is given by

$$\frac{y}{\theta}\bigg|_{\omega=\omega_g} = \frac{G_3^2}{K}\sqrt{\frac{K}{M}}\sqrt{\frac{MK}{G_1^2G_3^2}}\left(\frac{\omega_o^2}{\omega_g^2} - 1\right) \qquad (5.2.12)$$

Thus the ratio of the system gains at ω_o and ω_g is proportional to the square of the frequencies, proportional to the ratio of inertias, M and K, and inversely proportional to the squares of the generator constants.

It is also of interest to note that neither the circuit inductance nor the circuit resistance appears in Eq. 5.2.11 or 5.2.12. The explanations are that at ω_o the seismometer may be regarded as a high impedance generator, or constant current source (cf. paragraph 3.2.1), so that the current through the galvanometer is independent of circuit impedance, while at ω_g the galvanometer may be considered a high impedance load so that the current in the electrical circuit approaches zero.

By inspection it can be seen that if $G_1^2/M = G_3^2/K$ then Eq. 5.2.9 is unaltered as to form and numerical value by an interchange of free angular frequencies as might be obtained by adjustment of S and U leaving all other parameters unchanged. Considering the engineering difficulties associated with the production of long and short period galvanometers and seismometers, serious consideration should be given to this manner of interconnection for, if a satisfactory response-frequency curve can be obtained, then it may well be a matter of less engineering difficulty to make the natural frequency of the seismometer of the order of say 5 cps and the galvanometer 1 cps than vice versa.

Since the circuit will not permit the inclusion of attenuators for the control of sensitivity its use requires considerable dynamic range in the equipment used to determine the galvanometer deflection.

Using the parameters of an available Benioff Variable Reluctance Seismometer and a commercial galvanometer the authors have not been successful in determining circuit constants leading to adequate suppression of the resonant peaks, near ω_o and ω_g, in the amplitude-frequency response curve. They are of the opinion that the inductance associated with any variable reluctance transducer of reasonable rest-point stability and negligible mechanical damping precludes its use in a direct coupled system. The possibility of the judicious use of mechanical damping to suppress these peaks has not been investigated.

Section 6

SEISMOMETER-GALVANOMETER SYSTEMS WITH NEGLIGIBLE INDUCTANCE

6.1 General

Since seismometers using moving coil transducers can be built with very little inductance in the electrical circuit it is of interest to examine the form which the equations of motion take as $L \rightarrow 0$.

As in Section 5, the transient solutions of the equations of motion have not been obtained. However, some success has been obtained in describing the response-frequency curves for several special cases.

It may be noted that terms of the form $\sqrt{M/K}\ (\dot{y}/\theta)$ which appear throughout this report represent a normalized gain expressed in reciprocal form. The reciprocal form has been used since it permits expressing the frequency dependent factors in the equations for system response to simusoids in algebraic forms which are relatively simple.

6.2 Seismometer, Resistive T Pad, and Galvanometer

A seismometer having negligible inductance may be approximated by a second order linear differential equation so that it becomes permissible to speak of a "critical damping resistance" and a "ratio to critical damping" definable as

$$\mathcal{R}_o = G_1^2/2\sqrt{SM} = G_1^2/2\omega_o M = \omega_o G_1^2/2S$$

and λ_o equal to $\mathcal{R}_o$ divided by the circuit resistance seen by the seismometer. These definitions are consistent with those for a galvanometer introduced in Section 4 and 5 and are rigorous under the assumption of negligible damping other than that due to dissipation in the electrical circuit. It will also be convenient to use a mean ratio to critical damping, λ_m, defined as $\lambda_m \equiv \sqrt{\lambda_o \lambda_g}$, and the geometric mean of the two natural frequencies, ω_m, defined as $\omega_m \equiv \sqrt{\omega_o \omega_g}$.

Making the appropriate substitutions, letting $L \rightarrow 0$, equation 5.2.6 may be rewritten

$$2\alpha\lambda_m\omega_m\sqrt{\frac{M}{K}}\,\frac{d}{dt}\left(\frac{X}{M}-\frac{d^2y}{dt^2}\right) =$$

$$\left\{\left(\frac{d^2}{dt^2}+2\lambda_o\omega_o\frac{d}{dt}+\omega_o^2\right)\left(\frac{d^2}{dt^2}+2\lambda_g\omega_g\frac{d}{dt}+\omega_g^2\right)-4\alpha^2\lambda_m^2\omega_m^2\frac{d^2}{dt^2}\right\}\theta \qquad (6.2.1)$$

Letting $X = 0$ and $d/dt = j\omega$ Eq. 6.2.1 may be written as

$$2\alpha\lambda_m\omega_m\sqrt{\frac{M}{K}}\,\frac{y}{\theta} = 2\lambda_o\omega_o\,\frac{\omega_g^2-\omega^2}{\omega^2}+2\lambda_g\omega_g\,\frac{\omega_o^2-\omega^2}{\omega^2}$$

$$-\,j\omega\left[\frac{\omega_g^2-\omega^2}{\omega^2}\;\frac{\omega_o^2-\omega^2}{\omega^2}+4\,(\alpha^2-1)\,\lambda_m^2\,\frac{\omega_m^2}{\omega^2}\right] \qquad (6.2.2)$$

Earth displacement will be in phase (or 180° out of phase) with galvanometer coil displacement when ω satisfies the relation

$$2\omega^2 = \omega_g^2+\omega_o^2+4\lambda_m^2\,(1-\alpha^2)\,\omega_m^2 \pm \sqrt{[\omega_g^2+\omega_o^2+4\lambda_m^2\,(1-\alpha^2)\,\omega_m^2]^2-4\omega_m^4} \qquad (6.2.3)$$

In the special case of $\alpha = 1$ (direct coupling) the two values for ω in Eq. 6.2.3 are ω_o and ω_g. It can be shown that for $\lambda_m^2 > 0$ (which is the case for any real system containing only passive elements) and $\alpha < 1$ these two points lie outside the region between ω_o and ω_g.

Earth velocity and galvanometer displacement are in phase at

$$\omega_v^2 = \omega_m^2 \frac{\lambda_o\omega_g + \lambda_g\omega_o}{\lambda_g\omega_g + \lambda_o\omega_o} \qquad (6.2.4)$$

Equation 6.2.4 shows that this "quadrature" frequency is independent of α, must always lie between ω_o and ω_g, and for $\lambda_o = \lambda_g$ it is at ω_m.

Elementary considerations suggest that ω_v is that frequency at which the LC circuit equivalents of the seismometer and galvanometer show series resonance. This concept is discussed in detail in paragraph 7.2 and can be used in the calibration of a seismic system (see 9.3.5.3).

The general expression for system gain at the in-phase (and 180° out of phase) point is algebraically messy, involving, as it does, the insertion of the value of ω from Eq. 6.2.3 into the real part of Eq. 6.2.2. This statement is also true for the system gain at the quadrature point, the equations being 6.2.4 and the imaginary part of 6.2.2.

The gain at $\omega = \omega_o$ is given by

$$\alpha^2 \frac{\omega_o}{\omega_g}\frac{M}{K}\left|\frac{y}{\theta}\right|^2_{\omega=\omega_o} = \frac{\lambda_o}{\lambda_g}\left(\frac{\omega_g}{\omega_o} - \frac{\omega_o}{\omega_g}\right)^2 + 4\lambda_m^2(\alpha^2 - 1)^2 \qquad (6.2.5)$$

while at $\omega = \omega_g$ Eq. 6.2.2 leads to

$$\alpha^2 \frac{\omega_g}{\omega_o}\frac{M}{K}\left|\frac{y}{\theta}\right|^2_{\omega=\omega_g} = \frac{\lambda_g}{\lambda_o}\left(\frac{\omega_o}{\omega_g} - \frac{\omega_g}{\omega_o}\right)^2 + 4\lambda_m^2(\alpha^2 - 1)^2 \qquad (6.2.6)$$

For the special case of $\lambda_o = \lambda_g$ the ratio of these gains is the ratio of the natural frequencies, independent of the value of α and of the exact value of λ_o and λ_g.

As before it will be noted that, as the attenuation of the T pad increases, not only does the overall system gain decrease but the shapes of the response-frequency curves approach those of a pair of buffered filters. The possible response-frequency curves which may be obtained at small values of α for various values of ω_o, ω_g, λ_o and λ_g have been well considered in the literature. One special case is further discussed in subsection 6.4.

Also, as in the case of circuits containing inductance, it is not possible to design a T pad which will alter absolute gain without simultaneously altering the relative shape of the amplitude and phase-frequency response curves.

Admittedly, within engineering tolerances it is feasible to design an attenuator for any given seismometer and galvanometer which will give an adjustable system gain with reasonably stable response-frequency curves, provided α is not permitted to become too great.

Several features of Eq. 6.2.1 should be pointed out as being of some interest to the equipment designer. Having decided on values of ω_o, ω_g, λ_o and λ_g which will produce desired shapes for the response-frequency curves at low values of α and a maximum tolerable value for α, then the maximum overall system gain, or magnification depends only on M/K and the method used to record θ, the galvanometer deflection.

A second point of interest is the fact that the system response--both transient and steady state--is independent of whether ω_o is the natural frequency of the seismometer or galvanometer. That is, the designer is at liberty to select specific values for S, U, G_1, G_3 and the values of resistance in the T pad within the restrictions imposed by a prior selection of ω_o, ω_g, λ_o, λ_g, $\sqrt{M/K}$ and range of α. In this sense these systems differ from buffered electronic filters. For instance the following two systems are, as magic boxes, identical in the response of the galvanometer coil to earth motion or calibration techniques making use of a force on the bob.

System A	Seismometer	System B
1 kg	mass	1 kg
0.01 cps	natural frequency	1 cps
1000 ohm	critical damping resistance	100 ohm
50 ohm	coil resistance	50 ohm
any	T pad	same as for System A with input and output terminals interchanged

	Galvanometer	
1×10^{-7} kgm^2	moment of inertia	1×10^{-7} kgm^2
1 cps	natural frequency	0.01 cps
100 ohm	critical damping resistance	1000 ohm
50 ohm	coil resistance	50 ohm

Instrument designers will agree that, at least for vertical component devices, system B would probably be more easily fabricated as a practical, stable system.

6.3 Direct Coupled Seismometer-Galvanometer Systems

With the elimination of inductance from the equations of motion it becomes of interest to re-examine the direct coupled case ($R_s \rightarrow \infty$) for potentially useful response-frequency curves.

As $R_s \rightarrow \infty$ no point is served by attempting to distinguish R_1 from R_3. Throughout this section the series loop resistance will simply be called R. It should be noted that as $R_s \rightarrow \infty$ the relation $\lambda_o/\lambda_g = \mathcal{R}_o/\mathcal{R}_g$ holds.

Letting $\alpha = 1$, Eq. 6.2.2 may be rewritten

$$2\lambda_m\omega_m\sqrt{\frac{M}{K}}\,\frac{y}{\theta} = 2\lambda_o\omega_o\,\frac{\omega_g^2 - \omega^2}{\omega^2} + 2\lambda_g\omega_g\,\frac{\omega_o^2 - \omega^2}{\omega^2} \tag{6.3.1}$$

$$- j\omega\,\frac{\omega_g^2 - \omega^2}{\omega^2}\;\frac{\omega_o^2 - \omega^2}{\omega^2}$$

As mentioned in paragraph 5.2.2, y and θ are in phase (or 180° out of phase at ω_o and ω_g for all values of the series loop resistance while the quadrature frequency ω_v is given by Eq. 6.2.4.

The system gain for $\omega = \omega_o$ and $\omega = \omega_g$ may be written, from Eqs. 6.2.5 and 6.2.6

$$\frac{M}{K}\left|\frac{y}{\theta}\right|^2_{\omega = \omega_o} = \frac{\lambda_o\omega_g}{\lambda_g\omega_o}\left(\frac{\omega_g}{\omega_o} - \frac{\omega_o}{\omega_g}\right)^2 \tag{6.3.2}$$

and

$$\frac{M}{K}\left|\frac{y}{\theta}\right|^2_{\omega = \omega_g} = \frac{\lambda_g\omega_o}{\lambda_o\omega_g}\left(\frac{\omega_o}{\omega_g} - \frac{\omega_g}{\omega_o}\right)^2 \tag{6.3.3}$$

The ratio of the system gains at these two frequencies may be written

$$\left|\frac{\theta_{\omega_g}}{\theta_{\omega_o}}\right|_{y\ \text{constant}} = \frac{\lambda_o\omega_g}{\lambda_g\omega_o} \tag{6.3.4}$$

From Eq. 6.3.4 it can be seen that if $\lambda_o = \lambda_g$ then the system gain (amplitude basis) is proportional to the ratio of the two natural frequencies. On an earth velocity basis these two gains would be equal. In the direct coupled case, $\lambda_o = \lambda_g$ is equivalent to stating that the two instruments have the same critical damping resistances. This special case is considered at length in the following paragraph (6.3.1).

If $\lambda_o \omega_g = \lambda_g \omega_o$ then the system gain (amplitude basis) is the same at the two natural frequencies. That is, if the critical damping resistances of the instruments are in direct proportion to their natural frequencies then the system gain at these two frequencies is the same on an amplitude basis. This special case is discussed in paragraph 6.3.2.

A third possible special case of interest is that in which the quadrature frequency shall be the arithmetic mean of the in phase and 180° phase shift frequencies. This case is intimately related to the problem of dispersion which, in the present state of the art, is too elaborate a subject for inclusion in this paper.

6.3.1 Direct coupled system, "constant velocity" response.

As indicated above, when a seismometer and galvanometer have equal critical damping resistances then the system gains at the natural frequencies of the two instruments will be equal on an earth velocity basis.

Throughout this subsection it will be convenient to use the notation $\dot{y} = dy/dt = j\omega y$ for sinusoidal earth velocity excitation. Since $\lambda_o = \lambda_g = \lambda_m$ the subscript will be omitted.

It will also be convenient to use a normalized angular frequency ϕ defined as $\phi \equiv \omega/\omega_m$. ϕ therefore, is always positive, $\phi < 1$ for $\omega < \omega_m$; $\phi = 1$ at the geometric mean of the natural frequencies and $\phi > 1$ for $\omega > \omega_m$. From Eq. 6.2.2

$$\frac{1}{\omega_m}\sqrt{\frac{M}{K}}\,\frac{\dot{y}}{\theta} = \frac{1}{2\lambda}\left[\left(\phi + \frac{1}{\phi}\right)^2 - \left(\frac{\omega_g + \omega_o}{\omega_m}\right)^2\right] - j\left(\frac{\omega_g + \omega_o}{\omega_m}\right)\left(\phi - \frac{1}{\phi}\right)$$

or

$$\frac{1}{\omega_m}\sqrt{\frac{M}{K}}\,\frac{\dot{y}}{\theta} = \frac{1}{2\lambda}\left[\left(\phi - \frac{1}{\phi}\right)^2 - \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2\right] - j\left(\frac{\omega_g + \omega_o}{\omega_m}\right)\left(\phi - \frac{1}{\phi}\right) \qquad (6.3.5)$$

Inspection of Eq. 6.3.5 shows that: (a) The amplitude response, on an earth velocity basis, when plotted against log Φ (or log ω) is symmetrical about the point $\Phi = 1$ $(\omega = \omega_m)$, (b) At $\Phi = 1$ $(\omega = \omega_m)$ galvanometer displacement is in phase with earth velocity for all values of λ, (c) At $\Phi = 1$ $(\omega = \omega_m)$ the system gain varies directly with λ. Alternatively, for a given critical damping resistance, the system gain varies inversely with the loop resistance of the circuit.

Since Eqs. 6.3.2 and 6.3.3 show that the system gains at $\omega = \omega_o$ and $\omega = \omega_g$ are independent of λ (or the circuit loop resistance) while c) above indicates that the gain at the geometric mean frequency varies inversely with the circuit loop resistance, it follows that for high values of loop resistance $(\lambda \rightarrow 0)$ the amplitude response curve must have two maxima and one minimum (three extremes, all real). At low values of circuit resistance there is only one maximum (three extremes, one real, two imaginary). From all of the above reasoning, we see that there may be a unique value of λ representing a triple root, at $\Phi = 1$, for these extremes.

If one evaluates $d \left| \frac{\dot{y}}{\theta} \right|^2 / d\ln\Phi^2$ for Eq. 6.3.5 and sets the resulting function equal to zero, $\Phi^2 = 1$ will be a root corresponding to the extreme at $\omega = \omega_m$. The other two extremes occur at the roots of

$$(\Phi + \frac{1}{\Phi})^2 = (1 - 2\lambda^2)\left(\frac{\omega_g + \omega_o}{\omega_m}\right)^2$$

These two roots will also occur at $\Phi = 1$ when

$$\lambda^2 = \frac{1}{2}\left(\frac{\omega_g - \omega_o}{\omega_g + \omega_o}\right)^2 \qquad (6.3.6)$$

Note that as the two natural frequencies approach each other $\lambda \rightarrow 0$ while as they are more and more separated $\lambda \rightarrow 1/\sqrt{2}$. Substituting the value of λ from Eq. 6.3.6 into Eq. 6.3.5 one finds that the "midband" gain, at $\omega = \omega_m$, is given by

$$\sqrt{\frac{M}{K}} \left| \frac{\dot{y}}{\theta} \right|_{\omega_m} = \frac{\sqrt{2}}{2} \omega_m \left(\frac{\omega_g}{\omega_o} - \frac{\omega_o}{\omega_g}\right) , \tag{6.3.7}$$

while the system gains at the two natural frequencies are given by

$$\sqrt{\frac{M}{K}} \left| \frac{\dot{y}}{\theta} \right|_{\substack{\omega_o \ \& \\ \omega_g}} = \omega_m \left(\frac{\omega_g}{\omega_o} - \frac{\omega_o}{\omega_g}\right) . \tag{6.3.8}$$

Comparing Eqs. 6.3.7 and 6.3.8 it can be seen that the response at the two natural frequencies is down 3 db from the midband gain.

If Eq. 6.3.6 is used to eliminate λ from Eq. 6.3.5, one obtains

$$\frac{\sqrt{2}}{\omega_m} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \sqrt{\frac{M}{K}} \frac{\dot{y}}{\theta} = \left(\phi - \frac{1}{\phi}\right)^2 - \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 - j\sqrt{2} \left| \frac{\omega_g - \omega_o}{\omega_m} \right| \left(\phi - \frac{1}{\phi}\right) \tag{6.3.9a}$$

or, taking the square of the absolute value

$$\frac{2}{\omega_m^2} \left(\frac{\omega_g - \omega_o}{\omega_g + \omega_o}\right)^2 \frac{M}{K} \left| \frac{\dot{y}}{\theta} \right|^2 = \left(\phi - \frac{1}{\phi}\right)^4 + \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^4 \tag{6.3.9b}$$

The latter is convenient for computing amplitude frequency response curves.

Several curves for different values of ω_g/ω_o are shown in Fig. 6.1. The designer will see that using Fig. 6.1 he can quickly estimate the ratio of M/K required to have a desired earth velocity correspond to a given trace amplitude, once he has defined the band width and mechanism for recording galvanometer deflections. We have, for convenience, called this special case the "flat velocity fourth order direct-coupled case."

A discussion of Eq. 6.3.9 vis-a-vis its asymptotes as the octave separation of ω_o and ω_g changes is of interest for comparison with the response curve developed in subsection 6.4.

For $\phi \gg 1$ Eq. 6.3.9 is asymptotic to

$$\frac{\sqrt{2}}{\omega_m} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \sqrt{\frac{M}{K}} \left| \frac{\dot{y}}{\theta} \right| = \phi^2 \tag{6.3.10}$$

while for $\phi \ll 1$ the asymptote is

$$\frac{\sqrt{2}}{\omega_m} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \sqrt{\frac{M}{K}} \left| \frac{\dot{y}}{\theta} \right| = \phi^{-2} \tag{6.3.11}$$

The intersection of asymptotes is at $\phi = 1$ with an ordinate value

$$\omega_m \left| \frac{\theta}{\dot{y}} \right| \sqrt{\frac{K}{M}} = \sqrt{2} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \tag{6.3.12}$$

Parenthetically it will be noted that as $|\omega_g - \omega_o| \to \infty$ this intersection $\to \sqrt{2}$ while as $\omega_g/\omega_o \to 1$ this intersection $\to 0$.

It is also of interest to locate the intersections of the curve with the asymptotes. This is readily done as follows:

$$\text{Let } a \equiv \left| \frac{\omega_g - \omega_o}{\omega_m} \right| \tag{6.3.13}$$

a is therefore a measure of the octave separations of ω_o and ω_g.

Then Eq. 6.3.9b may be expanded to

$$\frac{2a^2}{a^2 + 4} \; \frac{M}{\omega_m^2 K} \left| \frac{\dot{y}}{\theta} \right|^2 = \phi^4 - 4\phi^2 + 6 + a^4 - \frac{4}{\phi^2} + \frac{1}{\phi^4}$$

While for the low frequency asymptote, Eq. 6.3.11, one may write

$$\frac{2a^2}{a^2 + 4} \; \frac{M}{\omega_m^2 K} \left| \frac{\dot{y}}{\theta} \right|^2 = \frac{1}{\phi^4}$$

so that the ratio to the low frequency asymptote may be written. The curve and its asymptote intersect at those values of ϕ for which this ratio is one, i.e. the intersections are at the roots of

$$\phi^2 \, [\phi^6 - 4\phi^4 + (6 + a^4)\,\phi^2 - 4] = 0$$

It will be noted that the bi-cubic in brackets, of this equation, is in a form suitable for graphically obtaining the real root from Fig. 3.1 with $\rho = 4$ and $\gamma^2 = a^4 + 5$. Doing so, one finds that as $a \to 0$, $\phi^2 \to 2$; for $a = 1$, $\phi^2 = 1$; and for $a > 1$, $\phi^2 < 1$. The situation is more clearly shown in the sketches of log-log plots in Fig. 6.2 (a), (b) and (c).

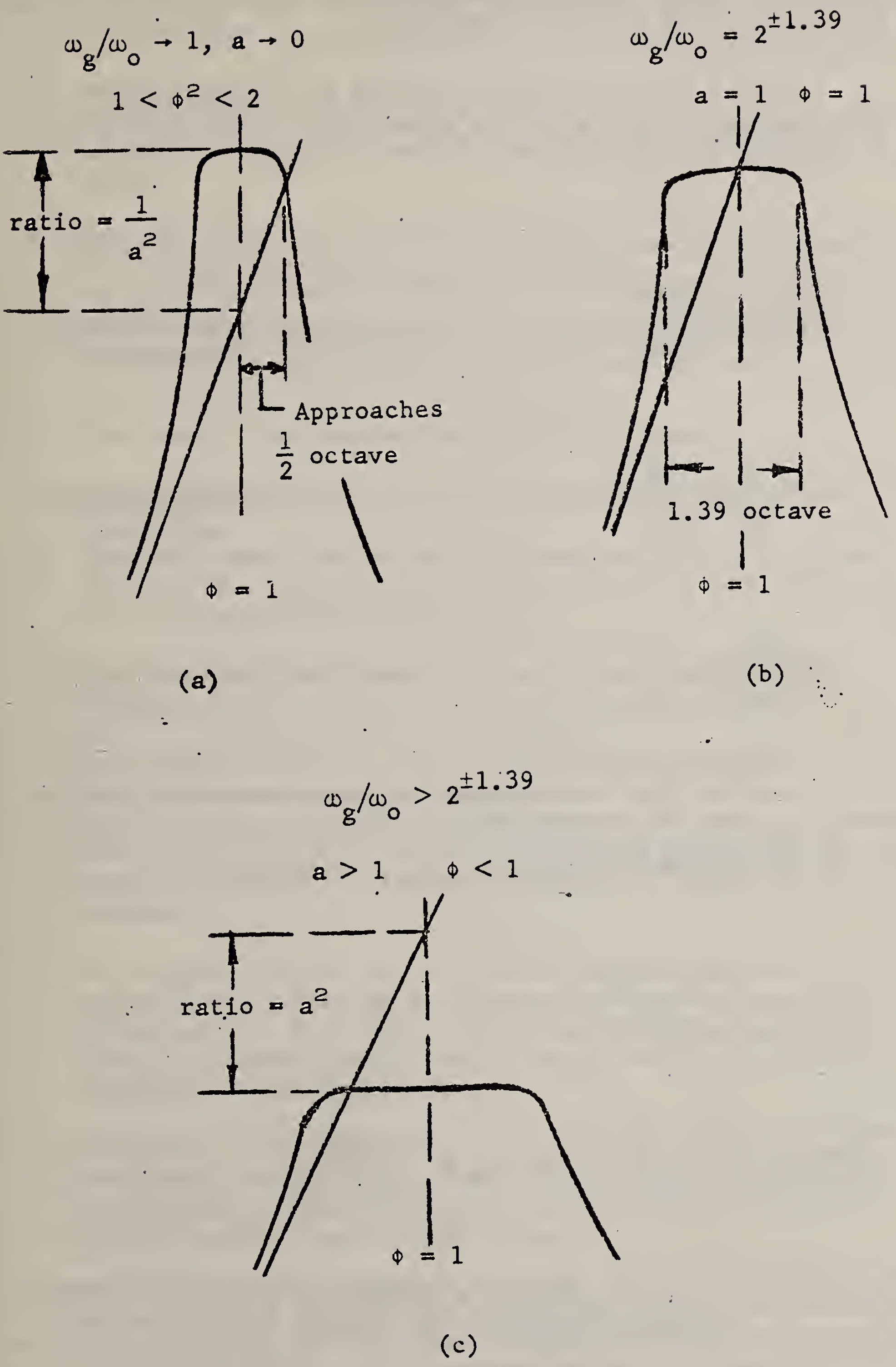

$\omega_g/\omega_o \rightarrow 1$, $a \rightarrow 0$
$1 < \phi^2 < 2$
ratio = $\frac{1}{a^2}$
Approaches $\frac{1}{2}$ octave
$\phi = 1$
(a)
$\omega_g/\omega_o = 2^{\pm 1.39}$
$a = 1$ $\phi = 1$
1.39 octave
$\phi = 1$
(b)
$\omega_g/\omega_o > 2^{\pm 1.39}$
$a > 1$ $\phi < 1$
ratio = a^2
$\phi = 1$
(c)

Figure 6.2

The frequencies of the intersections of the midband tangent with the asymptotes can be obtained from the relations:

$$\frac{M}{K}\left|\frac{\dot{y}}{\theta}\right|^2 = \left(\frac{a^2+4}{a^2}\right)\left[\left(\frac{\omega}{\omega_m}-\frac{\omega_m}{\omega}\right)^4 + a^4\right] \qquad \text{from Eq. 6.3.9}$$

$$= \left(\frac{a^2+4}{a^2}\right)\left(\frac{\omega}{\omega_m}\right)^4 \qquad \text{high frequency asymptote}$$

$$= \left(\frac{a^2+4}{a^2}\right)\left(\frac{\omega_m}{\omega}\right)^4 \qquad \text{low frequency asymptote}$$

$$= \left(\frac{a^2+4}{a^2}\right)(a)^4 \qquad \text{midband tangent}$$

Calling the intersection of the high frequency asymptote with the midband tangent ω_h, and the corresponding intersection of the midband tangent with the low frequency asymptote ω_ℓ one may write

$$\frac{\omega_h}{\omega_m} = a = \left|\frac{\omega_g - \omega_o}{\omega_m}\right| \quad \text{or} \quad \omega_h = |\omega_g - \omega_o| \qquad \text{and}$$

$$\frac{\omega_m}{\omega_\ell} = a = \left|\frac{\omega_g - \omega_o}{\omega_m}\right| \quad \text{or} \quad \omega_\ell = \left|\frac{\omega_g\omega_o}{\omega_g - \omega_o}\right| .$$

That is, the frequency at which the high frequency asymptote intersects the midband tangent is the difference of the natural frequencies while the intersection of the midband tangent with the low frequency asymptote occurs at a period equal to the difference of the natural periods of the instruments.

At this point it may be in order to summarize the properties and limitations of the flat velocity fourth order direct-coupled case.

1. When plotted on a log-log basis the galvanometer deflection for unit earth velocity will be symmetrical about the geometric mean of the two natural frequencies.

2. The response curve is convex upward between ω_o and ω_g having maximum "flatness" at the geometric mean of the two instrument natural frequencies. The first three derivatives of the gain are zero at $\omega = \omega_m$.

3. The asymptotes have slopes of 12 db/octave.

4. The 3 db points are at the two natural frequencies.

5. The knees are the sharpest obtainable, without introduction of a concave upward region in the midband, for a fourth order system.

6. The curve is independent of interchange of natural frequencies of the galvanometer and seismometer.

7. For a given value of M/K the greater bandwidth will have the lesser midband gain.

8. For a given value of M/K the midband gain will be greater than that obtainable with a T pad coupled system.

9. As no gain control can be placed between the seismometer and galvanometer it is necessary for any given pair of instruments that the recording mechanism be provided with dynamic range and gain controls adequate to the application.

10. The gain (angle/velocity) of the system, at constant bandwidth, can only be altered by changing the ratio of the two instrument inertias and is proportional to the square root of this ratio.

11. The working gain obviously depends on the means of recording the deflection of the galvanometer.

6.3.2 Direct coupled system, "constant amplitude" response

It was implied at the end of subsection 6.3 that, at least superficially, there seem to be conditions under which a close coupled seismometer-galvanometer system will approximate a constant amplitude response in the midband region. A few preliminary remarks on elementary grounds may be appropriate.

Well above its natural frequency a moving coil seismometer having negligible inductance has an emf proportional to velocity, and the problem in developing a "constant amplitude" system is so to adjust the interaction terms in the amplitude-frequency response equations as most nearly to correct the inherently rising characteristic with frequency of the seismometer. It may not be possible in the general case to arrive at an acceptable compromise. Another point suggesting difficulty is that for $\lambda_o \neq \lambda_g$ there is a range of values for the series resistance which results in one instrument being overdamped while the other is underdamped and this range becomes greater as the value of λ_o/λ_g departs from unity. Another point to remember is that the low frequency asymptote is 18 db/octave while the high is only 6 so that equations having the simplicity and symmetry of Eq. 6.3.9 cannot be expected.

The mathematical attack might proceed as follows: from Eq. 6.3.4 it follows that the system gain on an amplitude basis will be the same at the two natural frequencies provided the critical damping resistances of the two instruments are in the ratio of their natural frequencies. The in-phase and 180° phase shift frequencies are ω_o and ω_g while the quadrature frequency is, from Eq. 6.2.4,

$$\omega_v^2 = \frac{2\omega_o^2\omega_g^2}{\omega_o^2 + \omega_g^2} \qquad (6.3.14)$$

Substituting $\lambda_o\omega_g = \lambda_g\omega_o$ in Eq. 6.3.1 one may write

$$\sqrt{\frac{M}{K}}\,\frac{y}{\theta} = 2\,\frac{\omega_m}{\omega}\left(\frac{\omega_m}{\omega} - \frac{\omega}{\omega_m}\right) - \left(\frac{\omega_o - \omega_g}{\omega_m}\right)^2 - \frac{j}{2\lambda_m}\,\frac{\omega_m}{\omega}\left[\left(\frac{\omega_m}{\omega} - \frac{\omega}{\omega_m}\right)^2 - \left(\frac{\omega_o - \omega_g}{\omega_m}\right)^2\right]$$

or

$$\sqrt{\frac{M}{K}}\,\frac{y}{\theta} = \frac{2\omega_m^2}{\omega_v^2}\left(\frac{\omega_v^2}{\omega^2} - 1\right) - \frac{j\omega}{2\lambda_m\omega_m}\left(1 - \frac{\omega_o^2}{\omega^2}\right)\left(1 - \frac{\omega_g^2}{\omega^2}\right) \qquad (6.3.15)$$

For prescribed values of ω_o and ω_g, Eq. 6.3.15 represents a single infinity of phase-frequency curves all of whose members have their quadrant points at $\omega = 0$, $\omega = \omega_o$, $\omega = \omega_v$, $\omega = \omega_g$ and $\omega = \infty$. The amplitude-frequency curves of this infinity all pass through the points given by

$$\left|\frac{\theta}{y}\right|_{\omega_o \;\&\; \omega_g} = \frac{\omega_o\omega_g}{\omega_g^2 - \omega_o^2}\sqrt{\frac{M}{K}} \qquad (6.3.16)$$

Inspection of Eq. 6.3.15 shows that if λ_m be small then the gain in the neighborhood of ω_v will be small compared to that at ω_o and ω_g. Conversely, as λ_m is increased the relative gain in the neighborhood of ω_v increases. That is, at least for small values of λ_m, the amplitude-frequency response curve will have three extremes (two maxima and one minimum). One may attempt arbitrarily to define the "constant amplitude system" as having that amplitude-frequency response curve for which these three extremes coincide in frequency, i.e., that value of λ_m (if any real value exists) for which the amplitude-frequency curve is always convex upward on a log-log plot but most nearly linear in the mid-band region.

Examination of Eq. 6.3.15 for extremes is messy; however, for small value of λ_m one would expect two inflection points, lying between ω_o and ω_g, in the amplitude-frequency response curve. Further, the "constant amplitude system" might be that for which the two inflection points occur at the same frequency, and this attack on Eq. 6.3.15 is algebraically manageable. One obtains

$$\omega^4\left[\left(\frac{\omega_o}{\omega_g}+\frac{\omega_g}{\omega_o}\right)^2 + 2 - 16\lambda_m^2\left(\frac{\omega_o}{\omega_g}+\frac{\omega_g}{\omega_o}\right)\right] + 3\omega^2\left[16\lambda_m^2\omega_o\omega_g - 2(\omega_o^2+\omega_g^2)\right] + 6\omega_o^2\omega_g^2 = 0 \qquad (6.3.17)$$

The two roots of ω^2 in Eq. 6.3.17 give the frequencies of these inflection points for various values of λ_m, and examination shows that a single, real value of λ_m exists only if $\omega_o^2/\omega_g^2 + \omega_g^2/\omega_o^2 = 4$. This corresponds to a separation of ω_o and ω_g of 0.949 octave with $\lambda_m^2 = \sqrt{6}/24$ or $\lambda_m \simeq 0.319$. For a closer spacing of the two natural frequencies there are two real and distinct values of λ_m meeting the prescribed conditions while for greater separations of the natural frequencies all values of λ_m are complex. Figure 6.3 has been computed for the special case of 0.949 octave separation. The reader will agree that this curve leaves something to be desired as a "constant amplitude" response. The physical significance of a complex λ as required for instruments with greater separation of their natural frequencies escapes the authors. From this we may conclude that a broad-band "constant amplitude" response based on $\mathcal{R}_o/\mathcal{R}_g = \omega_o/\omega_g$ is not obtainable in a close-coupled system free of mechanical damping. The reader will note that the mathematical difficulties encountered above are quite consistent with the remarks at the beginning of this subsection.

A more general line of attack is as follows.

For convenience let $2(\lambda_o\omega_g + \lambda_g\omega_o) \equiv A\omega_m$ and $2(\lambda_o\omega_o + \lambda_g\omega_g) \equiv B\omega_m$. Incidently A/B is, from Eq. 6.2.4, ω_v^2/ω_m^2. Then Eq. 6.3.1 may be written

$$2\lambda_m\sqrt{\frac{M}{K}}\,\frac{y}{\theta} = \frac{\omega_m^2}{\omega^2}A - B - j\,\frac{\omega}{\omega_m}\left(\frac{\omega_m^4}{\omega^4} - \frac{\omega_g^2+\omega_o^2}{\omega^2} + 1\right) \qquad (6.3.18)$$

ϕ has already been defined as ω/ω_m so the absolute value of earth displacement over galvanometer deflection may be written

$$4\lambda_m^2 \frac{M}{K} \left| \frac{y}{\theta} \right|^2 = \frac{1}{\phi^6} + \left[A^2 - 2\left(\frac{\omega_g^2 + \omega_o^2}{\omega_m^2} \right) \right] \frac{1}{\phi^4} + \left[2 - 2AB + \left(\frac{\omega_g^2 + \omega_o^2}{\omega_m^2} \right)^2 \right] \frac{1}{\phi^2} \tag{6.3.19}$$

$$+ B^2 - 2\,\frac{\omega_g^2 + \omega_o^2}{\omega_m^2} + \phi^2$$

Note that the coefficients of the second and third terms on the right hand side will be zero when

$$A^2\omega_m^2 \equiv 4\,(\lambda_o\omega_g + \lambda_g\omega_o)^2 = 2\,(\omega_g^2 + \omega_o^2) \quad \text{and} \tag{6.3.20}$$

$$2AB \equiv 8\,(\lambda_o\omega_g + \lambda_g\omega_o)\,(\lambda_o\omega_o + \lambda_g\omega_g)/\omega_m^2 = 2 + \left(\frac{\omega_g^2 + \omega_o^2}{\omega_m^2} \right)^2 \tag{6.3.21}$$

Explicit solutions of Eqs. 6.3.20 and 6.3.21 for λ_o and λ_g are messy. As given they may readily be used for numerical evaluation of λ_o, λ_g and λ_m, for any given bandwidth. Figure 6.4 shows the values of these quantities plotted against ω_o/ω_g for physically realizable systems.

When λ_o and λ_g have values consistent with Eqs. 6.3.20 and 6.3.21 then Eq. 6.3.19 takes the form

$$4\lambda_m^2 \frac{M}{K} \left| \frac{y}{\theta} \right|^2 = \frac{1}{\phi^6} + B^2 - A^2 + \phi^2 \qquad (6.3.22)$$

where $A^2 = 2\,(\omega_g^2 + \omega_o^2)/\omega_m^2$ and $B = (8 + A^4)/8A$. This equation is plotted in Fig. 6.5 for 1, 2 and 4 octave separations of the instrument's natural frequencies. ω_v^2 may be obtained as

$$\frac{\omega_v^2}{\omega_m^2} = \frac{4\omega_m^2\,(\omega_g^2 + \omega_o^2)}{2\omega_m^4 + (\omega_g^2 + \omega_o^2)^2}\,. \qquad (6.3.23)$$

As $(\omega_g - \omega_o) \to \infty$; $\omega_v/\omega_m \to 2\omega_o/\omega_m$ and as $\omega_g \to \omega_o$; $\omega_v/\omega_m \to 2/\sqrt{3}$ which is $> \omega_o/\omega_m$ or ω_g/ω_m, i.e. the quadrant point crosses an in-phase point. As such a crossing is not physically realizable, the minimum ratio of ω_g/ω_o (or ω_o/ω_g) which may be used is $4\sqrt{3}$ or 0.396 octave separation.

It will be noted that for a physically realizable system the right side of Eq. 6.3.22 must be positive for all values of ϕ. The minimum value of $1/\phi^6 + \phi^2$ occurs at $\phi = \sqrt[8]{3}$ and is $4\sqrt[4]{3}/3$. Therefore $3\,(B^2 - A^2) + 4\sqrt[4]{3} > 0$ is necessary. Substituting for A and B the values indicated above one can show that this restriction is equivalent to the requirement that the instrument natural frequencies be separated by at least 0.396 octave, the same restriction as is imposed by quadrant point frequency considerations.

Analytical examination of the expressions for λ_o, λ_g and λ_m shows that they are all real and positive for any bandwidth greater than 0.396 octave.

Incidently $\phi = \sqrt[3]{3}$ is the frequency at which the system gain is a maximum, independent of bandwidth. Also the ratio of instrument critical damping resistances is given by

$$\frac{\mathcal{R}_o}{\mathcal{R}_g} = \frac{\lambda_o}{\lambda_g} = \frac{A\omega_g - B\omega_o}{B\omega_g - A\omega_o} \qquad (6.3.24)$$

6.3.3 Direct coupled system, "constant acceleration" response

The technique exemplified by Eqs. 6.3.18 through 6.3.24 suggests an elegant procedure for developing a "constant acceleration" system.

Let A and B have such values that the third and fourth terms on the right hand side of Eq. 6.3.19 are zero. Then, since $|\ddot{y}/\omega_m^2| = \phi^2 |y|$, an equation analogous to Eq. 6.3.22 may be written,

$$4 \frac{\lambda_m^2}{\omega_m^4} \frac{M}{K} \left| \frac{\ddot{y}}{\theta} \right|^2 = \phi^6 + A^2 - B^2 + \frac{1}{\phi^2} \qquad (6.3.25)$$

where

$$B^2 = 2 \frac{\omega_g^2 + \omega_o^2}{\omega_m^2} \qquad (6.3.26)$$

and

$$2AB = 2 + \left(\frac{\omega_g^2 + \omega_o^2}{\omega_m^2} \right)^2 \quad \text{as before} \qquad (6.3.27$$

Again explicit solutions for λ_o and λ_g are messy.

ω_v, the quadrant frequency, is given by

$$\frac{\omega_v^2}{\omega_m^2} = \frac{2\omega_m^4 + (\omega_g^2 + \omega_o^2)^2}{4\omega_m^2 (\omega_g^2 + \omega_o^2)} \qquad (6.3.28)$$

Comparing Eqs. 6.3.28 and 6.3.23 it will be noted that the normalized quadrant frequencies are reciprocals.

It can be shown that the system is physically realizable provided that ω_g/ω_o or ω_o/ω_g is greater than $\sqrt[4]{3}$, i.e. the two instrument natural frequencies are separated by more than 0.396 octave.

Considering the two lines of attack on possible "constant amplitude" midband response characteristics given in paragraph 6.3.2 the "constant velocity" case considered in paragraph 6.3.1, and the "constant acceleration" case above, it appears that one may design a system showing a midband response which is either rising or falling with increasing frequency, by a proper choice of the ratio of the two critical damping resistances. Further, having set λ_o/λ_g to provide a desired overall response then the proper choice of series loop resistance for the electrical circuit will result in a mean ratio to critical damping, λ_m, giving any desired amount of "saddle" or "droop" in the midband response. One may therefore, within limits, adjust the system response for optimum detection capability for seismic disturbances in terms of the spectral distribution of local background noise by proper selection of these parameters. Additional flexibility in the tailoring of response-frequency characteristics is available as shown in Section 7.

6.4 Further Remarks on the T Pad Problem

While the flat velocity, direct coupled response characteristic developed in Section 6.3.1 produces the highest possible gain for a given ratio of seismic mass to galvanometer moment of inertia, it suffers from the disadvantage that no conveniently adjustable gain control may be placed between the two instruments. That is, its full capabilities require that each instrument pair be designed for its intended location with a prior knowledge of the noise background obtaining at that location. While this requirement is, in principle, not severe, in practice certain difficulties are encountered. The noise background is often not known in advance and further, the hard facts of economics suggest that a certain degree of instrument standardization is desirable. Hence the remainder of this section is devoted to the following hypothetical but practical problem.

A seismometer and galvanometer are available which will have the desired amplitude-frequency response when used as a flat velocity fourth order direct coupled system. The sensitivity with a certain recording system is adequate to the quietest proposed location. The dynamic range of the galvanometer-recording system is 40 db between noise level and excessive non-linearity. For any given set-up the operating dynamic range (smallest to largest useful excursion) is 20 db. The seismometer and galvanometer can be replicated within adequate tolerance. The question is, what is the minimum alteration of frequency response which can be obtained with the insertion of fixed T pads having attenuations which are multiples of, say, 20 db, in order to use replicates at more noisy locations?

For a T pad with ~ 20 db attenuation, $\alpha = 0.1$ and $\alpha^2 \ll 1$ so that Eq. 6.2.1, with $X = 0$ and $d/dt = j\omega$, may be written

$$2\alpha\lambda_m\omega_m\omega^2\sqrt{\frac{M}{K}}\,\frac{\dot{y}}{\theta} = (\omega_o^2 - \omega^2 + 2j\lambda_o\omega_o\omega)\,(\omega_g^2 - \omega^2 + 2j\lambda_g\omega_g\omega) \tag{6.4.1}$$

Recalling the remark following Eq. 6.2.6, let the values of the resistors in the T pad be chosen so that $\lambda_o = \lambda_g = \lambda$ and using $\phi = \omega/\omega_m$, Eq. 6.4.1 may be put in the form

$$4\alpha^2 \frac{\lambda^2}{\omega_m^2}\frac{M}{K}\left|\frac{\dot{y}}{\theta}\right|^2 = \left(\phi - \frac{1}{\phi}\right)^4 + \left(\phi - \frac{1}{\phi}\right)^2 \left[4\lambda^2\left(\frac{\omega_g + \omega_o}{\omega_m}\right)^2\right.$$

$$\left. - 2\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 - 8\lambda^2\right] + \left[\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 + 4\lambda^2\right]^2 \qquad (6.4.2)$$

So that if the attenuator be designed to have λ such that

$$4\lambda^2\left(\frac{\omega_g + \omega_o}{\omega_m}\right)^2 - 2\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 - 8\lambda^2 = 0$$

i.e.

$$2\lambda^2 = \frac{(\omega_g - \omega_o)^2}{\omega_g^2 + \omega_o^2} \qquad (6.4.3)$$

(note the similarity to Eq. 6.3.6)

Then Eq. 6.4.2 may be written

$$\frac{2\alpha^2}{\omega_g^2 + \omega_o^2}\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 \frac{M}{K}\left|\frac{\dot{Y}}{\theta}\right|^2 = \left(\phi - \frac{1}{\phi}\right)^4$$

$$+\left[\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 + 2\,\frac{(\omega_g - \omega_o)^2}{\omega_g^2 + \omega_o^2}\right]^2 \qquad (6.4.4)$$

This response curve may, for convenience, be called the "flat velocity fourth order loose coupled case." On comparison with the flat velocity fourth order direct coupled case, Section 6.3.1, it can be seen that the frequency function is the same but the constants are different (Eq. 6.3.9 compared to Eq. 6.4.4) and the resulting frequency response curve will, as a result, be altered by more than a multiplicative constant.

First, in the matter of λ, several points may be brought out. In the direct coupled case it is necessary that the critical damping resistances of the two instruments be the same, while for the T pad case it is necessary that the resistors of the T pad be adjusted so that the instruments are operated at the same fraction of critical damping. In so far as T pads are more easily fabricated, adjusted or altered than the magnet structures of galvanometers or seismometers, the T pad linked system is easier to build.

The optimum value of λ depends only upon the octave separation of the natural frequencies of the two instruments. For both the direct coupled and the T pad arrangements $\lambda \to 0$ as $|\omega_g - \omega_o| \to 0$ and $\lambda \to 1/\sqrt{2}$ as $|\omega_g - \omega_o| \to \infty$, while for intermediate values of $|\omega_g - \omega_o|$ the optimum value of λ will be slightly higher for the T pad system than for the direct coupled system.

For a given value of earth velocity, calling the galvanometer deflection for the direct coupled system θ_d and that for the T pad system θ_T, one may write the ratio of the midband gains as

$$\left(\alpha \frac{\theta_d}{\theta_T}\right)^2 = \frac{\left(\omega_g + \omega_o\right)^2}{\omega_g^2 + \omega_o^2} \qquad (6.4.5)$$

by dividing Eq. 6.4.4 by Eq. 6.3.9b.

Equation 6.4.5 shows that, unless the octave separation of the natural frequencies is large, the midband attenuation is not given by α. With regard to the hypothetical question with which this section was introduced it follows that for prescribed midband attenuations to be obtained by the insertion of T pads, Eq. 6.4.5 must be used to obtain the appropriate values of α to define the resistor values required to fabricate the T pads.

Second, as regards the 3 db points, it has been shown by Eqs. 6.3.7 and 6.3.8 that the 3 db points for the direct coupled case are at ω_o and ω_g. The 3 db points for the T pad arrangement are given by

$$(\phi - \frac{1}{\phi})^2_{3\ \text{db}} = (\frac{\omega_g - \omega_o}{\omega_m})^2 \ (1 + 2 \frac{\omega_m^2}{\omega_g^2 + \omega_o^2}) \qquad (6.4.6)$$

While algebraic solution of Eq. 6.4.6 is messy, its numerical evaluation for specific values of ω_o and ω_g is relatively easy. As compared to the direct coupled system response, a T pad containing system will have a wider passband. Table 6.1 gives some typical values showing that the fractional increase in bandwidth for the T pad arrangement is greater for narrow band $(\omega_o/\omega_g \rightarrow 1)$ systems.

Table 6.1

Computed 3 db Points for Direct and T Pad Linked Fourth Order Flat Velocity Systems

Natural frequencies of instruments and 3 db points direct coupled		Octave separation	3 db points when T pad coupled	
$\frac{\omega_o}{\omega_m}$	$\frac{\omega_g}{\omega_m}$		$\frac{\omega}{\omega_m}$	$\frac{\omega}{\omega_m}$
0.707	1.414	1	0.632	1.58
0.500	2.000	2	0.443	2.26
0.250	4.000	4	0.237	4.21
0.125	8.000	6	0.123	8.12

Section 7

SPECIAL RESPONSE-FREQUENCY CHARACTERISTICS

7.1 General

In the course of this work several systems have been studied which lead to differential equations of orders higher than the fifth. Of these only two have yielded results of sufficient interest to justify their inclusion in this report. The first, which we have chosen to call the "Notch Filter", appears to be useful for suppressing a narrow band of high amplitude noise within, or adjacent to, an otherwise interesting spectral region. Depending on the number of such notches included, the resulting equations may become of very high order. The utility and operation of a notch filter are discussed chiefly from elementary considerations. The equations of motion of such a system are also pertinent to the question of using several galvanometers with a single seismometer, to record simultaneously two separate spectral regions which may or may not overlap. The second system of interest is characterized by a sixth order differential equation leading to overall response curves which we have chosen to call "the flat velocity sixth", "the flat amplitude sixth" and "the flat acceleration sixth". They appear to be useful in working with a seismometer system having very high gain, and, as will be shown (subsection 7.4) are well adapted to use with telemetering techniques.

7.2 The Notch Filter

Suppose one wishes to set up a broadband, high gain, seismic system to record information in the frequency range of say 0.03 to 3 cps. For the system to have high gain and yet keep the total weight of the seismometer manageable one may elect to use one of the direct coupled arrangements of Section 6. Unfortunately, one knows a priori, or quickly learns by experiment, that the desired instrument location has a noise background consisting largely of microseisms of, say, 6 second period whose amplitude is such that they completely dominate the record. One might consider sensing the galvanometer coil position electronically and include within the electronic portion of the system a suitable filter. While this would eliminate the 6 second energy from the record it would still be present in the earlier portions of the system and could be great enough severely to affect the dynamic range of the overall system. It would therefore be desirable to eliminate this energy as early in the system as possible. This leads to the thought that a conventional LC trap might be inserted in the electrical loop connecting the seismometer and galvanometer.

Considerations of resonant frequency, impedance level, circuit resistance, and desired Q suggest that the inductor and its associated condenser bank required to produce a suitable conventional LC trap would be difficult to obtain, costly and bulky. Referring to Fig. 4.1 it would appear that, considered as a two terminal network, a galvanometer having negligible mechanical damping and a period of 6 seconds might well serve as the desired LC trap. Since, as the mechanical damping of a galvanometer is decreased, R_θ increases, it follows from equation 4.3.6 that the attenuation at the natural frequency of the added galvanometer may, in principle, be made as great as we please. That is, the amplitude response curve of the system would have a deep notch centered on the natural frequency of the added galvanometer.

As to the width of the notch, a detailed mathematical analysis is messy, however approximations suitable for preliminary calculations may be developed as follows:

Calling the natural frequency of the added galvanometer ω_N and its critical damping resistance $\mathcal{R}_N$, we see that when ω_o and ω_g are far removed from ω_N we may approximate the system by the circuit of Fig. 7.1 wherein the total circuit resistance is taken as R; the portion to the right of the dotted line being the electrical equivalent of the added galvanometer.

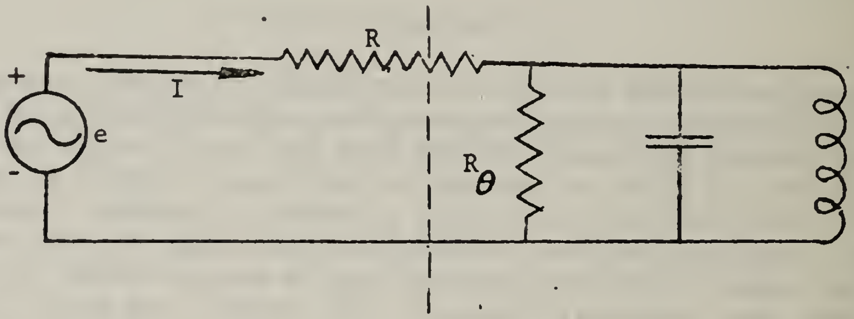


Figure 7.1

From equation 4.3.5 we may write

$$\frac{e}{i} = R + R_\theta \frac{1 + j\frac{R_\theta}{\mathcal{R}_N}\left(\frac{\omega_N^2 - \omega^2}{2\omega_N\omega}\right)}{1 + \frac{R_\theta^2}{\mathcal{R}_N^2}\left(\frac{\omega_N^2 - \omega^2}{2\omega_N\omega}\right)^2} \tag{7.2.1}$$

or

$$\left|\frac{e}{i}\right|^2 = R^2 + (2R + R_\theta)\,\frac{4\omega_N^2\omega^2\mathcal{R}_N^2 R_\theta}{4\omega_N^2\omega^2\mathcal{R}_N^2 + R_\theta^2(\omega_N^2 - \omega^2)^2} \tag{7.2.2}$$

For values of ω well removed from ω_N equation 7.2.2 approaches $|e/i| \to R$. The 3 db points at the sides of the notch will occur at those values of ω for which $|e/i|^2 = 2R^2$. Combining this with equation 7.2.2 gives

$$\left(\frac{1}{\lambda_N}\right)^2 = \left(\frac{R}{\mathcal{R}_N}\right)^2 = \frac{2\frac{R}{R_\theta} + 1}{\frac{\mathcal{R}_N^2}{R_\theta^2} + \left(\frac{\omega_N^2 - \omega_{3\text{ db}}^2}{2\omega_N\omega_{3\text{ db}}}\right)^2} \tag{7.2.3}$$

In many air damped galvanometers of period shorter than one second, the mechanical damping may be neglected. Then $\mathcal{R}_N \ll R_\theta \gg R$, and equation 7.2.3 simplifies to

$$2\lambda_N = \left|\frac{\omega_N}{\omega_{3\text{ db}}} - \frac{\omega_{3\text{ db}}}{\omega_N}\right| \text{ or } \frac{\omega_N}{\omega_{3\text{ db}}} = \sqrt{\lambda_N^2 + 1} \pm \lambda_N \tag{7.2.4}$$

Table 7.1 shows the notch width, separation of 3 db points, for several values of λ_N such as might be obtained by using a galvanometer with a wide range of adjustment for $\mathcal{R}_N$, such as a galvanometer with adjustable flux.

Table 7.1

Notch Width as a Function of Filter Damping

(For galvanometer having negligible internal losses)

λ_N	3 db points as	ω/ω_N	Approx octave separation of 3 db points
1/4	0.781	1.281	0.8
$\sqrt{2}/4$	0.707	1.414	1
1/2	0.618	1.618	1.3
$\sqrt{2}/2$	0.518	1.932	1.9
1	0.414	2.414	2.9
$\sqrt{2}$	0.318	3.146	5
2	0.236	4.236	9

Having designed a system consisting of a seismometer and galvanometer, direct coupled, according to a desired amplitude-frequency response curve as outlined in Section 6 and having computed a notch filter as indicated above, it is of interest to insert these parameters into the equations of motion for a direct-coupled system which includes a notch filter in the electrical loop. Such a system is diagrammed in Figure 7.2.

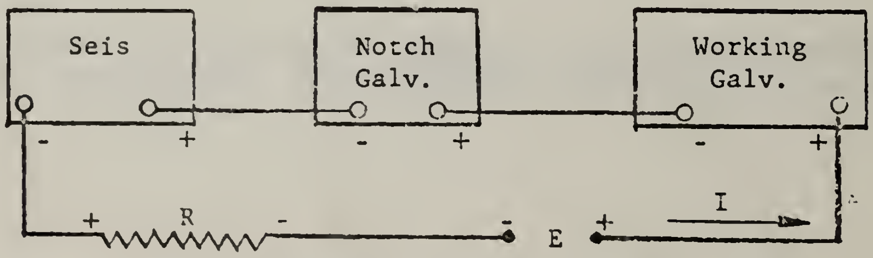


Figure 7.2

Since the electrical circuit is series connected, we may lump the various coil resistances and any added circuit resistance as the single resistor R. Using the subscript N to denote the notch galvanometer and W to denote the working galvanometer, the basic equations of motion, with mechanical damping neglected, may be written

$$X - M\frac{d^2y}{dt^2} = M\frac{d^2x}{dt^2} + Sx - G_1 I \qquad (7.2.5)$$

$$0 = K_W\frac{d^2\theta_W}{dt^2} + U_W\theta_W - G_W I \qquad (7.2.6)$$

$$0 = K_N\frac{d^2\theta_N}{dt^2} + U_N\theta_N - G_N I \qquad (7.2.7)$$

$$E = G_1\frac{dx}{dt} + G_W\frac{d\theta_W}{dt} + G_N\frac{d\theta_N}{dt} + RI \qquad (7.2.8)$$

These four equations may be used to eliminate x, I and θ_N. For $X = 0$ (no force applied directly to the bob), $E = 0$ (no external EMF applied to the circuit), $d/dt = j\omega$ (sinusoidal excitation), remembering that $\omega_{()} = \sqrt{U_{()}/K_{()}}$ and defining $\lambda_{()}$ as $\mathcal{R}_{()}/R$ where $\mathcal{R}_{()} = G^2_{()}/2\omega_{()}K_{()}$ the resulting equation may be written

$$2\omega^2 \sqrt{\omega_o \omega_W} \sqrt{\lambda_o \lambda_W} \sqrt{\frac{M}{K_W}} \frac{\dot{y}}{\theta_w} = (\omega_o^2 - \omega^2)(\omega_W^2 - \omega^2)$$

$$+ 2j\omega \left[(\lambda_W \omega_o + \lambda_o \omega_W)\, \omega_o \omega_W - (\lambda_W \omega_W + \lambda_o \omega_o)\omega^2 \right] \tag{7.2.9}$$

$$+ 2j\omega\, (\omega_o^2 - \omega^2)(\omega_W^2 - \omega^2) \frac{\lambda_N \omega_N}{(\omega_N^2 - \omega^2)}$$

It is of interest to note that the parameters of the notch filter appear only in the final term and also that algebraic manipulation of equation 6.3.1 will put it into precisely the form of equation 7.2.9 except, of course, for the absence of this final term.

Returning to the supposition with which this subsection was opened, the curve marked with circles of Fig. 7.3 is the response curve for a flat velocit fourth order direct-coupled system according to equation 6.3.9 with $\omega_g = 0.06\pi$ and $\omega_o = 6\pi$. From equation 7.2.4 a notch filter centered on 6 seconds with the 3 db points of the notch approximating 3 and 12 seconds corresponds to $\omega_N = 0.333\pi$ and $\lambda_N = 0.750$. Using $\lambda_o = \lambda_W = \lambda_m = 0.693$ from equation 6.3.6 together with the above values for the other parameters, equation 7.2.9 yields the curve marked with triangles on Fig. 7.3. The differences represent the errors which result from first approximation design. In any real case the errors introduced by ignoring the mechanical losses, especially in a low frequency galvanometer, may represent a greater deviation of the resulting amplitude-frequency response curve than suggested by Fig. 7.3 and any final design should take account of the mechanical losses by their inclusion in an equation appropriate to the special case at hand.

Reexamining equation 7.2.9 it can be seen that as ω_N approaches closely to either ω_o or ω_g the value of the last term oscillates violently for small changes of ω in this region. In particular for $\omega_o < \omega_N < \omega_W$ and for $\omega_o < \omega < \omega_N$ the real term on the right side of equation 7.2.9 may be very small and the first imaginary term will be positive while the second is negative. There is one value of ω for which the two imaginary terms cancel so that the system gain becomes quite large - possibly much greater than at any other value of ω. A similar situation arises when the notch filter frequency is adjacent to, but outside the frequency region bounded by ω_o and ω_g.

On elementary grounds the existence of these maxima may be explained as follows. From the electrical equivalent circuit any of these instruments shows capacitative reactance above its natural frequency and inductive reactance below its natural frequency so that, when two or more are series connected, there is always at least one frequency at which series resonance occurs. For a single galvanometer and seismometer this has been shown to be ω_v. From this line of reasoning it follows that maxima of this type may be suppressed at the expense of lessened attenuation in the notch by shunting the notch galvanometer with a resistor. In so far as the coil resistance of the notch galvanometer is small compared to the total circuit loop resistance such an added resistor is operationally indistinguishable from R_θ for the notch galvanometer.

This line of reasoning shows that the use of two galvanometers in the same electrical loop to broaden the rejection band is impractical since at some intermediate value of frequency the system gain will tend to rise to substantially the same value as it would have without the added instruments.

Such behavior, in a real system would, generally speaking, be undesirable. This instability of equation 7.2.9 indicates that in this region the response of a real system would be dominated by factors ignored in equation 7.2.9, namely the mechanical losses in the two instruments having proximate natural frequencies.

An alternative solution to the problem of spurious resonance between instruments having proximate natural frequencies is to tolerate the increased mass required and use a T pad connection between the seismometer and working galvanometer. A notch filter may then be safely placed in that arm of the T not containing the instrument of proximate natural frequency, provided $\alpha \ll 1$.

7.3 Multiple Frequency Response Systems

Having inserted a second galvanometer into a direct-coupled system to reject energy which would normally appear in the working galvanometer it is reasonable to ask whether it is practical to record the motions of the notch galvanometer coil as a means of monitoring the rejected frequency band. On elementary grounds it would appear that the frequency response curve of the seismometer-notch galvanometer combination would be the difference between the "circle" and "triangle" curves of Fig. 7.3. This would be true if, 1) the addition of the notch galvanometer did not alter, at any frequency, the impedance seen by the seismometer, 2) there were no interaction between galvanometers and 3) the seismometer pier may be considered as a truly infinite impedance source. The third assumption seems reasonable; in a direct-coupled system the first two are not.

Equation 7.2.9 takes account of the interactions among the three instruments. As far as a record of the excursions of the notch galvanometer coil is concerned the working galvanometer may be considered as a notch filter on the notch galvanometer. Therefore equation 7.2.9 may be used to compute the amplitude frequency response of the seismometer-notch galvanometer by interchanging the parameters assigned to the working and notch galvanometers. This has been done for the particular system discussed in Section 7.2 and plotted as the "square" curve of Fig. 7.3.

Since a notch galvanometer will generally be added to reject relatively high level signals there is no requirement for a small moment of inertia in the notch galvanometer. In fact the reverse may be true, as a relatively large moment of inertia may be required, since throughout this discussion it has been an essential though implicit assumption that the notch galvanometer perform as a linear element.

Extension of the above discussion to a system consisting of, say, a one second seismometer series-connected to a 60 second galvanometer and a 5 cps galvanometer simultaneously is fairly obvious.

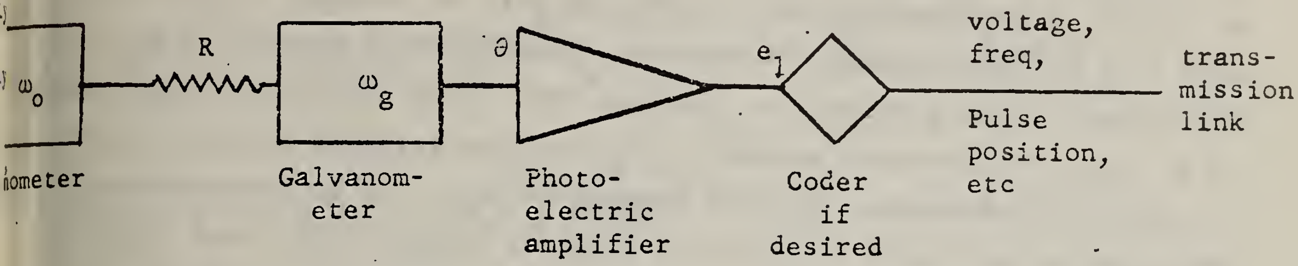
The possibilities associated with a system of two galvanometers each having added series resistance, parallel connected to a single seismometer appear interesting. The appropriate equations have not been attempted; however their development, at least for special cases, appears straightforward in the light of the material presented in Section 6.

7.4 The Telemetering Problem

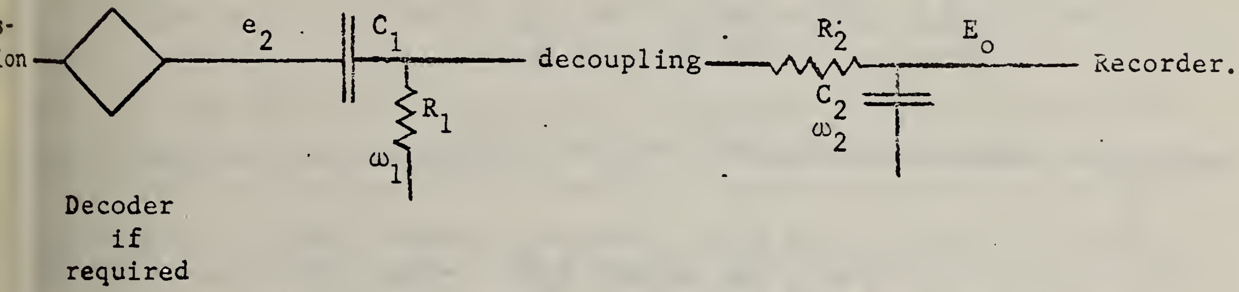
The operation of a seismometer at high gain may require that it be located at a distance from cultural noise. If the position of the galvanometer coil be sensed electrically then it becomes feasible to use conventional telemetering techniques to transmit the intelligence over great distances. Conventional telemetering techniques frequently include filters to restrict the passband of the circuit and thereby reduce the effect of noise generated in the circuit. Conventionally these filters are designed to provide a passband only broad enough for the purpose at hand since the effective noise of the circuit is reduced by limiting the passband. Since the presence of this filter does alter the system frequency response the question may be asked as to whether one can take advantage of this fact and so modify the response of the seismometer-galvanometer combination that, when combined with filters inserted just ahead of the final recorder, the overall system has a desired response-frequency characteristic. To show a general line of attack we will set up the development of the sixth order flat velocity direct-coupled system in paragraph 7.4.1 and in paragraph 7.4.2 show the alterations of circuit necessary to produce a "flat amplitude sixth" and a "flat acceleration sixth".

7.4.1 A "flat velocity sixth" order response

The block diagram for a potentially useful seismometer-galvanometer system is shown in Fig. 7.4.



Equipment at the seismometer vault



Equipment at the recording site

Figure 7.4

The following assumptions will be made with respect to the block diagram of Fig. 7.4.

The seismometer has negligible inductance and mechanical losses, and the symbols previously assigned will be used. R is the total resistance of the electrical circuit connecting the seismometer and galvanometer. It includes the coil resistance of the two instruments, connecting cables and any additional resistance which may be deliberately added. Since the galvanometer is presumed to have negligible inductance and mechanical loss, the symbols assigned and used in Sections 4, 5 and 6 will be used here. Electrical sensing of the galvanometer coil position is here presumed to be accomplished by a photoelectric amplifier which produces a voltage strictly proportional to coil position and independent of intelligence frequency. It will be presumed to follow the relation $e_1 = k\theta$[1]. The coder-decoder combination will be presumed to have unity gain[2] at all intelligence frequencies. The high pass element C_1R_1 will be assigned a time constant τ_1 so that the 3 db point for this element is at ω_1. The low pass element R_2C_2 will be assigned a time constant τ_2 giving a 3 db point for this element at ω_2. It is presumed that some type of unity gain isolation is provided between C_1R_1 and R_2C_2. It is further presumed that the recorder is not frequency selective and represents no load on the low-pass filter. The equation relating the output voltage E_o to the galvanometer coil position is

$$\frac{de_2}{dt} = k\frac{d\theta}{dt} = \frac{1}{\omega_2}\left(\frac{d}{dt} + \omega_2\right)\left(\frac{d}{dt} + \omega_1\right)E_o \tag{7.4.1}$$

which, for sinusoidal excitation, may be written

$$\theta = \left[\frac{\omega_1}{\omega_2} + 1 + j\left(\frac{\omega}{\omega_2} - \frac{\omega_1}{\omega}\right)\right]\frac{E_o}{k} \tag{7.4.2}$$

If the time constants of the high and low-pass filters be set so that $\omega_1\omega_2 = \omega_m^2$ then the signal attenuation due to the electronic portion of the system will be the same at ω_o and ω_g. With this restriction, the overall system will have the same gain on a velocity basis at ω_o and ω_g when $\Omega_o = \Omega_g$ or $\lambda_o = \lambda_g = \lambda_m$. As a matter of convenience the subscript of λ will be suppressed in this subsection. Therefore the over-all system response will be given by using equation 7.4.2 to eliminate θ from equation 6.3.5 to give

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[1] For practical instruments of this type k may be 500 to 1000 volts per radian.

[2] Available gains may range higher; the authors have used systems with an overall gain of five. Low level voltage controlled oscillators have been built which would give gains as high as 1000 with practical decoders.

$$\frac{k}{\omega_m}\sqrt{\frac{\omega_2}{\omega_1}}\sqrt{\frac{M}{K}}\frac{\dot{y}}{E_o} = \frac{1}{2\lambda}\left(\frac{\omega_1+\omega_2}{\omega_m}\right)\left[\left(\phi-\frac{1}{\phi}\right)^2 - \left(\frac{\omega_g-\omega_o}{\omega_m}\right)^2\right]$$

$$+\left(\frac{\omega_g+\omega_o}{\omega_m}\right)\left(\phi-\frac{1}{\phi}\right)^2 + j\left(\phi-\frac{1}{\phi}\right)\left\{\frac{1}{2\lambda}\left[\left(\phi-\frac{1}{\phi}\right)^2 - \left(\frac{\omega_g-\omega_o}{\omega_m}\right)^2\right]\right.$$

$$\left. - \left(\frac{\omega_1+\omega_2}{\omega_m}\right)\left(\frac{\omega_g+\omega_o}{\omega_m}\right)\right\} \qquad (7.4.3)$$

For any given values of ω_o and ω_g equation 7.4.3 represents a double infinity of curves depending on the octave separation of ω_1 and ω_2 and the value of the circuit resistance in the loop connecting the seismometer and galvanometer, i.e. λ. We wish to select that value of λ and those values of ω_1 and ω_2 (consistent with $\omega_1\omega_2 = \omega_m^2$) yielding the flattest midband response. This is equivalent to finding the curve having the highest order tangent to a horizontal straight line at $\omega = \omega_m$. This is equivalent to finding those values for λ and $(\omega_1 + \omega_2)/\omega_m$ which will make the first n derivatives of the amplitude-frequency response curve zero, where n is as large as possible.

The algebraic symbolism will be simplified by letting $c = k^2\omega_2 M\dot{y}^2/\omega_m^2\omega_1 K E_o^2$; $b = (\omega_1 + \omega_2)/\omega_m$ and $a^2 = (\omega_g - \omega_o)^2/\omega_m^2$ in writing the expression for the absolute value of the gain

$$4\lambda^2 c = \left(\phi-\frac{1}{\phi}\right)^6 + \left(\phi-\frac{1}{\phi}\right)^4\left[b^2 - 2a^2 + 4\lambda^2(a^2+4)\right]$$

$$+\left(\phi-\frac{1}{\phi}\right)^2\left[a^2(a^2-2b^2) + 4\lambda^2 b^2(a^2+4)\right] + a^4 b^2 \qquad (7.4.4)$$

The first and second derivatives of c with respect to $(\phi - \frac{1}{\phi})^2$ are both zero at $\phi = 1$ when

$$a^2 (a^2 - 2b^2) + 4\lambda^2 b^2 (a^2 + 4) = 0 \qquad (7.4.5)$$

and

$$b^2 - 2a^2 + 4\lambda^2 (a^2 + 4) = 0 \qquad (7.4.6)$$

respectively.

Solving equations 7.4.5 and 7.4.6 simultaneously for λ and b gives

$$2\omega_1 = |\omega_g - \omega_o| \mp \sqrt{(\omega_g - \omega_o)^2 - 4\omega_m^2} \qquad (7.4.7)$$

$$2\omega_2 = |\omega_g - \omega_o| \pm \sqrt{(\omega_g - \omega_o)^2 - 4\omega_m^2} \qquad (7.4.8)$$

and

$$\lambda = \frac{1}{2} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \qquad (7.4.9)$$

When the loop resistance R and the filters $(C_1R_1$ and $C_2R_2)$ are adjusted according to the upper sign in equations 7.4.7, 7.4.8 and 7.4.9 then equation 7.4.3 takes the forms

$$\frac{k}{\omega_m} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \sqrt{\frac{\omega_2}{\omega_1}} \sqrt{\frac{M}{K}} \frac{\dot{y}}{E_o} = \left| \frac{\omega_g - \omega_o}{\omega_m} \right| \left\{ 2 (\phi - \frac{1}{\phi})^2 - (\frac{\omega_g - \omega_o}{\omega_m})^2 \right\}$$

$$+ j (\phi - \frac{1}{\phi}) \left\{ (\phi - \frac{1}{\phi})^2 - 2 (\frac{\omega_g - \omega_o}{\omega_m})^2 \right\} \qquad (7.4.10)$$

and

$$\frac{k^2}{\omega_m^2}\left(\frac{\omega_g - \omega_o}{\omega_g + \omega_o}\right)^2 \frac{\omega_2 M}{\omega_1 K}\left|\frac{\dot{y}}{E_o}\right|^2 = \left(\phi - \frac{1}{\phi}\right)^6 + \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^6 \qquad (7.4.11)$$

Examination of equations 7.4.7 and 7.4.8 shows that: 1) The 3 db points of the individual RC filters are within the pass-band defined by ω_o and ω_g for all values of ω_o and ω_g and they approach ω_o and ω_g as the pass-band is widened. 2) The use of the lower sign in each equation represents the "crossed filter" setting of these two filters. The over-all system gain is reduced without any increase in system dynamic range, except for the trivial case where the system dynamic range is limited by the recorder. 3) There is a minimum separation of ω_o and ω_g for real values of ω_1 and ω_2. This separation is $\omega_g/\omega_o = 3 \pm 2\sqrt{2}$ or 2.542 octaves. For narrower bandwidths the values of ω_1 and ω_2 given by equations 7.4.7 and 7.4.8 are imaginary, i.e. the two RC filters must be replaced by a single LC filter of prescribed Q.

From equation 7.4.9 it can be seen that, as in the fourth order direct-coupled flat velocity case (eq 6.3.6), as $\omega_g/\omega_o \rightarrow 1$, $\lambda \rightarrow 0$ while the limiting value of λ is 1/2 critical as the pass-band is broadened. This limit is lower than for the corresponding fourth order case where λ may approach 0.707 critical.

The system midband gain is (from eq 7.4.10 or 7.4.11 with $\phi = 1$)

$$\left|\frac{E_o}{\dot{y}}\right|_{\omega_m} = \sqrt{\frac{M}{K}}\,\frac{k}{\omega_1}\left(\frac{\omega_m^2}{\omega_g^2 - \omega_o^2}\right)\left(\frac{\omega_m}{\omega_g - \omega_o}\right) \qquad (7.4.12)$$

while at $\omega = \omega_g$ and $\omega = \omega_o$ the system gain is

$$\left|\frac{E_o}{\dot{y}}\right|_{\omega_o,\ \omega_g} = \frac{1}{\sqrt{2}}\sqrt{\frac{M}{K}}\,\frac{k}{\omega_1}\left(\frac{\omega_m^2}{\omega_g^2 - \omega_o^2}\right)\left(\frac{\omega_m}{\omega_g - \omega_o}\right) \qquad (7.4.13)$$

Therefore the sixth order flat velocity direct-coupled system resembles the fourth order case in that the 3 db points for the overall system gain are at ω_o and ω_g.

Having determined the requirements for the sixth order flat velocity system it is of interest to note the gain, as observed at the galvanometer mirror, at ω_o, midband, and ω_g. Inserting the value of λ given by equation 7.4.9 back into equation 6.3.5 one finds that

$$\left|\frac{\theta}{\dot{y}}\right|_{\omega_o} = \left|\frac{\theta}{\dot{y}}\right|_{\omega_m} = \left|\frac{\theta}{\dot{y}}\right|_{\omega_g} = \sqrt{\frac{M}{K}}\left(\frac{\omega_m}{\omega_g^2 - \omega_o^2}\right) \qquad (7.4.14)$$

He is led to conclude that a devious method has been used to find those values for the time constants of a CR-RC filter which will make it 3 db down at ω_o and ω_g vis-a-vis ω_m (equations 7.4.7 and 7.4.8) and that value of λ, or circuit loop resistance which yields a midband droop of 3 db vis-a-vis the fourth order direct coupled flat velocity system, cf. equation 6.3.7 vis-a-vis equation 7.4.14.

As a matter of interest one may readily compute the quadrant points of the phase-frequency response curve for the sixth order direct-coupled flat velocity system. Taking ψ as the angle by which, for sinusoidal excitation, the voltage at the recorder leads the earth velocity we may write, from equation 7.4.10,

$$\tan\psi = \frac{\left(\phi - \frac{1}{\phi}\right)\left[2\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2 - \left(\phi - \frac{1}{\phi}\right)^2\right]}{\left|\frac{\omega_g - \omega_o}{\omega_m}\right|\left[2\left(\phi - \frac{1}{\phi}\right)^2 - \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2\right]} \qquad (7.4.15)$$

where $\omega_g - \omega_o$ in the denominator is always to be taken positive. The results are given in Table 7.2 from which it can be seen that there is a total phase shift of the output of three quadrants across the pass-band, as compared with the two quadrant shift characteristic of the fourth order case.

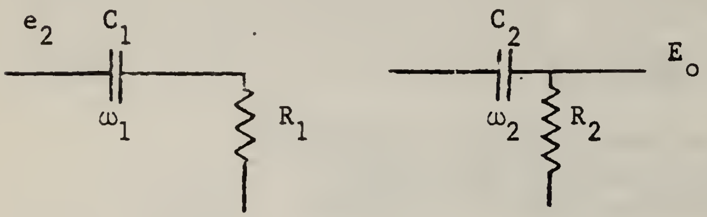
Table 7.2

Phase Lead of the Recorder Voltage with Respect to Earth Velocity for the Sixth Order Direct-Coupled Constant Velocity System

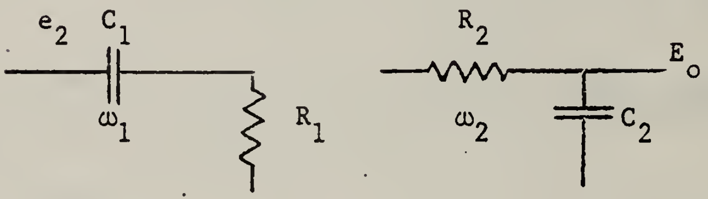
ω radians/sec	electrical degrees
$\to 0$	270
$(\sqrt{2}/2)\left(\sqrt{\omega_g^2 + \omega_o^2} - \lvert\omega_g - \omega_o\rvert\right)$	180
ω_o	135
$(\sqrt{2}/4)\left(\sqrt{(\omega_g + \omega_o)^2 + 4\omega_m^2} - \lvert\omega_g - \omega_o\rvert\right)$	90
ω_m	0
$(\sqrt{2}/4)\left(\sqrt{(\omega_g + \omega_o)^2 + 4\omega_m^2} + \lvert\omega_g - \omega_o\rvert\right)$	- 90
ω_g	- 135
$(\sqrt{2}/2)\left(\sqrt{\omega_g^2 + \omega_o^2} + \lvert\omega_g - \omega_o\rvert\right)$	- 180
$\to \infty$	- 270

7.4.2 A "Flat Amplitude" and a "Flat Acceleration Sixth" order response

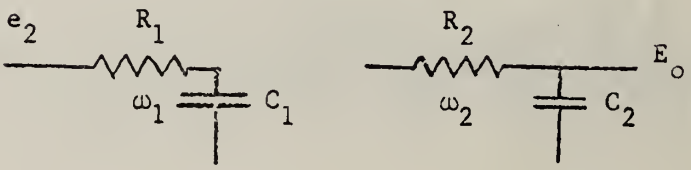
Consider a pair of decoupled RC filters. These may be arranged in three ways giving three slightly different differential equations



$$\frac{d^2 e_2}{dt^2} = \left(\frac{d}{dt} + \omega_1\right) \left(\frac{d}{dt} + \omega_2\right) E_o \tag{7.4.16}$$



$$\omega_2 \frac{de_2}{dt} = \left(\frac{d}{dt} + \omega_1\right) \left(\frac{d}{dt} + \omega_2\right) E_o \tag{7.4.17}$$



$$\omega_1 \omega_2 e_2 = \left(\frac{d}{dt} + \omega_1\right) \left(\frac{d}{dt} + \omega_2\right) E_o \tag{7.4.18}$$

In section 7.4.1 use was made of equation 7.4.17 with the relation $e_2 = k\theta$. If in Fig. 7.4 one interchanges the position of R_1 and C_1 then equation 7.4.18 applies. A derivation paralleling, in detail, that leading to equation 7.4.3 will yield equation 7.4.20. Alternatively the reader will note that the quantity $\frac{de_2}{dt}$ in equation 7.4.17 is replaced by $\omega_1 e_2$ in equation 7.4.18. If one replaces $\dot{y}$ in equation 7.4.3 by $\omega_1 y$ one obtains

$$\frac{k}{\omega_m}\sqrt{\frac{\omega_2}{\omega_1}}\sqrt{\frac{M}{K}}\,\frac{\omega_1 y}{E_o} = k\sqrt{\frac{M}{K}}\,\frac{y}{E_o} = \text{right hand side of eq. 7.4.3} \tag{7.4.19}$$

Then under the restrictions of equations 7.4.7, 7.4.8 and 7.4.9 on ω_1, ω_2 and λ, equations 7.4.10 and 7.4.11 become

$$k\left|\frac{\omega_g - \omega_o}{\omega_g + \omega_o}\right|\sqrt{\frac{M}{K}}\,\frac{y}{E_o} = \left|\frac{\omega_g - \omega_o}{\omega_m}\right|\left\{2\left(\phi - \frac{1}{\phi}\right)^2 - \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2\right\} + j\left(\phi - \frac{1}{\phi}\right)\left\{\left(\phi - \frac{1}{\phi}\right)^2 - 2\left(\frac{\omega_g - \omega_o}{\omega_m}\right)^2\right\} \tag{7.4.20}$$

and

$$k^2\left(\frac{\omega_g - \omega_o}{\omega_g + \omega_o}\right)^2\frac{M}{K}\left|\frac{y}{E_o}\right|^2 = \left(\phi - \frac{1}{\phi}\right)^6 + \left(\frac{\omega_g - \omega_o}{\omega_m}\right)^6 \tag{7.4.21}$$

The reader will note that the frequency variable function (right side) is the same in equation 7.4.10 and 7.4.20. The gain variable in equation 7.4.10 is earth velocity while in equation 7.4.20 the gain variable is earth amplitude. That is, we now have a "flat amplitude sixth order response" (eq. 7.4.20). The phase-frequency relation for equation 7.4.10 (Table 7.1) is changed for equation 7.4.20 only in that the reference phase is earth displacement rather than earth velocity. The designer will note that the interchange of R_1 and C_1 in Fig. 7.4 removes from the system all high pass filters other than that associated with the mechanical portions of the seismometer and galvanometer. In particular the displacements on the record will include drift of the galvanometer rest-point and of the electronic equipment as well as d-c offsets, if any. He may, fortunately, insert high pass filters as required provided their time constants are so long compared to the longest period of interest that they have negligible effect on the amplitude and phase response in the pass-band.

Turning our attention to an interchange of R_2 and C_2 in Fig. 7.4 rather than R_1 and C_1 as was done above then we must use equation 7.4.16 rather than 7.4.17 or 7.4.18. Doing this, equation 7.4.3 becomes

$$\frac{k}{\omega_2 \omega_m} \sqrt{\frac{\omega_2}{\omega_1}} \sqrt{\frac{M}{K}} \frac{\ddot{y}}{E_o} = \frac{k}{\omega_m^2} \sqrt{\frac{M}{K}} \frac{\ddot{y}}{E_o} = \text{right side of eq. 7.4.3} \qquad (7.4.22)$$

The equations corresponding to 7.4.10 and 7.4.11 may be written

$$\frac{k}{\omega_m^2} \left| \frac{\omega_g - \omega_o}{\omega_g + \omega_o} \right| \sqrt{\frac{M}{K}} \frac{\ddot{y}}{E_o} = \left| \frac{\omega_g - \omega_o}{\omega_m} \right| \left\{ 2(\phi - \frac{1}{\phi})^2 - (\frac{\omega_g - \omega_o}{\omega_m})^2 \right\}$$

$$+ j\,(\phi - \frac{1}{\phi}) \left\{ (\phi - \frac{1}{\phi})^2 - 2\,(\frac{\omega_g - \omega_o}{\omega_m})^2 \right\} \qquad (7.4.23)$$

and

$$\frac{k^2}{\omega_m^4} (\frac{\omega_g - \omega_o}{\omega_g + \omega_o})^2 \frac{M}{K} \left| \frac{\ddot{y}}{E_o} \right|^2 = (\phi - \frac{1}{\phi})^6 + (\frac{\omega_g - \omega_o}{\omega_m})^6 \qquad (7.4.24)$$

The reader will note that the frequency variable function (right side) is the same in equation 7.4.10 and 7.4.23. The gain variable in equation 7.4.10 is earth velocity while in equation 7.4.22 the gain variable is earth acceleration. That is, we now have a "flat acceleration sixth order response" (equation 7.4.24). The phase-frequency relation for equation 7.4.10 (Table 7.1) is changed for equation 7.4.23 only in that the reference phase is earth acceleration rather than earth velocity.

The designer will note that the interchange of R_2 and C_2 in Fig. 7.4 removes from the system all low pass filters other than that associated with the mechanical portions of the seismometer and galvanometer. In particular the displacements on the record will include the telemetry carrier, line pick-up, hum, electronic equipment noise, etc. He may, fortunately, insert low pass filters as required provided their 3 db points are at frequencies so high compared to the highest frequency of interest that they have negligible effect on the amplitude and phase response in the pass-band.

Section 8

SENSITIVITY LIMITS IN PRINCIPLE AND IN PRACTICE

8.1 General

In this section we shall discuss some effects introduced by the extraneous variables to which any real (as distinguished from ideal) seismic system is subjected in the course of "normal" use. Throughout this section it will be presumed that one is attempting to detect the smallest possible motions of the seismometer pier under the most difficult of field conditions. It will also be presumed that electronic means are used to sense the angular displacement of the galvanometer mirror and to relay this information to a recording location. The limitations in principle are considered with respect to idealized instruments, particularly in that each is presumed to have one and only one degree of mechanical freedom.

8.2 Brownian Motion

Brownian motion, or the thermal agitation of the movable portions of the mechanical system is not ordinarily thought of as being great enough to represent a sensitivity limitation on real systems. It does however represent a noise source which cannot be reduced by refinements of design or construction and therefore represents a limitation *in principle* on the maximum useful magnification of a system.

It will be presumed that the seismometer, the galvanometer, and the interconnecting cables are all at some constant temperature, T, degrees Kelvin. Any mechanical system has an average kinetic energy associated with it amounting to $kT/2$ for each degree of freedom, where k is Boltzmann's constant. This is the inherent thermal energy of the device and actually defines its temperature. The spring-mass combination of the seismometer or galvanometer has one degree of mechanical freedom[1] with which is associated an average kinetic energy of $kT/2$, an equal average potential energy and therefore an average total energy of kT. The potential energy considered here is that increment of energy stored in the spring due to the displacements of the mass from its *long-time average* position. The kinetic energy is that energy stored as velocity of the bob with respect to the frame in its excursions in the neighborhood of its long time average position. Any persistent change in these energies may properly be said to be a change in the temperature of the spring-mass system.

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[1] We are here explicitly ignoring modes of motion generally classed as spurious resonances.

A point which has sometimes been overlooked in the literature but which seems quite clear from the above remarks is that the coupling of any other system, which is at the same temperature, to a spring-mass combination cannot alter the long time average of either the potential or kinetic energy of the spring-mass system. If there were any change in the long time average of these energies it would represent a permanent displacement of thermal energy between two systems initially at the same temperature, in clear violation of the second law of thermo-dynamics.

Specifically the Johnson noise in the electrical circuit arises from the thermal agitation of the electrons in the conductors. When the resistors are at the same temperature as the mechanical components, there can be no net flow of energy between the electrical and mechanical degrees of freedom. Therefore the mean energy of thermal agitation of the bob of the seismometer and of the armature of the galvanometer is neither increased nor decreased by connection to the electrical system.

We do not mean to imply that the spectral distribution of the noise amplitude of a spring-mass system is independent of its coupling to other systems - it almost certainly is not - but the rms value of the Brownian motion cannot be altered by coupling between systems which are at the same temperature.

Proceeding on this basis it becomes a very simple matter to compute, quantitatively, as a function of frequency, that rms value of sinusoidal earth motion which will produce galvanometer excursions whose rms value is equal to the rms value of the Brownian motion of the galvanometer coil.

Each of the amplitude-frequency response curves relating seismometer excitation and galvanometer response which has been developed may be put in the form

$$\frac{1}{\omega_m^2} \frac{M}{K} \left| \frac{\dot{y}}{\dot{\theta}} \right|^2 = F(\omega) \qquad (8.2.1)$$

where F is a dimensionless function of frequency and the constants of the system.

For a system in which the final record gives equal weight to equal galvanometer velocities then eq. 8.2.1 may be rewritten as

$$\frac{\frac{1}{2} M |\dot{y}|^2}{\frac{1}{2} K |\dot{\theta}|^2} = \frac{\omega_m^2}{\omega^2} F \qquad (8.2.2)$$

The left side of equation 8.2.2 is in the form of the ratio of two kinetic energies. The denominator is the kinetic energy of the galvanometer coil resulting from a sinusoidal earth velocity, $\dot{y}$, at a frequency ω. If, for the denominator, we substitude the mean kinetic energy of Brownian motion we get

$$M\dot{y}^2 = kT \frac{\omega_m^2}{\omega^2} F \qquad (8.2.3)$$

from which, assuming a value for the absolute temperature T, one may calculate, at various frequencies, that sinusoidal earth velocity which will produce the same mean kinetic energy in the galvanometer as the average value due to Brownian motion.

On the other hand, for a recording system which gives equal weight to equal galvanometer displacements then equation 8.2.1 may be written in the form

$$\frac{\frac{1}{2} M \dot{y}^2}{\frac{1}{2} U \theta^2} = \frac{\omega_o}{\omega_g} F \qquad (8.2.4)$$

so that

$$M\dot{y}^2 = kT \frac{\omega_o}{\omega_g} F \qquad (8.2.5)$$

is the appropriate form for computing that value of sinusoidal earth velocity which will produce the same mean potential energy in the galvanometer as the average value due the Brownian motion.

The difference between equation 8.2.3 and 8.2.5 is related to the frequency distribution of Brownian motion energy.

It is of interest to note that the various amplitude-frequency response curves which have been developed have been quite independent. both as to form and as to system gain, of whether the seismometer or the galvanometer had the higher natural frequency Equation 8 2.5 shows that as regards sensitivity limited by Brownian motion one can do better, for a given inertial mass, by using a high frequency galvanometer and low frequency seismometer rather than vice versa.

We leave it to the reader to decide what ratio of signal to Brownian motion "noise" represents a "just detectable signal" as well as how to arrange the recording system so as not to inject noise of effective amplitude greater than that due to the Brownian motion of the galvanometer coil.[2]

An interesting corollary of equation 8.2.3, obvious as soon as it is pointed out, is that, given a recording mechanism adequate to record the Brownian motion of any galvanometer, the minimum detectable earth motion is independent of the moment of inertia of the galvanometer used. The availability of such a recording mechanism would ease the problems of the galvanometer designer considerably.

Another point which is somewhat obvious is that, while equations 8.2.3 and 8.2.5 have been developed assuming that the useful sensitivity of a proposed system is Brownian-motion limited, corresponding equations can be written for any less sensitive recording system by inserting the appropriate value of galvanometer rotation. We also see that by doubling the useful sensitivity of the recording system one may reduce by a factor of four the mass required to record a given earth motion. While this may be trivial in situations requiring only a few grams of inertial mass the same is not true when tens of kilograms are required.

Returning to the relative merits of T pad and direct coupling between the two instruments consider two systems based on the same galvanometer and recording mechanism, one containing a T pad and the other direct coupled. Assuming that the over-all response curves of the two systems are approximately the same then, for a factor of ten attenuation of the trace amplitude due to the T pad, an increase in the inertial mass by a factor of 100 is required to obtain earth motion sensitivity with the T pad system comparable to that of the direct coupled system. For high gain systems where the minimum mass dictated by Brownian motion considerations is on the order of a kilogram this factor weighs heavily in favor of direct coupling.

8.3 Quantum Effects

Seismic systems can be devised with such calculated gains as to give reasonable trace amplitudes for pier displacements of atomic dimensions. The question may be raised as to whether quantum considerations with their concept of finite energy increment represent a limitation, in principle, of system performance.

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[2] On the latter point it should be mentioned that there are commercially available galvanometer-photoelectric assemblies whose equivalent input noise computes rather closely to the expected Brownian motion limit.

Three such possibilities come to mind: (a) The application of the Heisenberg uncertainty principle to the position of the galvanometer, seismometer, or pier. This sets a limit to the product of the uncertainty in the coordinate and the uncertainty in the momentum. As applied to a galvanometer this may be written

$$(\Delta\theta) \quad (\Delta\ K\omega\theta) \simeq h/2\pi \sim 10^{-34} \text{ joule sec.}$$

If $K \sim 10^{-4}$ gr cm^2 or 10^{-11} kg m^2

and the highest frequency of interest be $\omega = 10^3/\text{sec}$

then $10^{-8}\ (\Delta\theta)^2 \sim 10^{-34}$

$\Delta\theta \sim 10^{-13}$ radians, a value much smaller than can be observed in the present state of the art. For a seismometer of 1 kg mass, one obtains $\Delta x \sim 3\text{x}10^{-18}$ meters, again far beyond present measurement techniques. For the pier the mass is larger and the corresponding displacement smaller. (b) One might be concerned as to whether energy is transmitted from the earth to the seismometer and galvanometer in quantum packets of observable size. (c) There is a possibility that the "noise" background is quantized in observable steps. In either of these cases the quantum step, of the order of $h\upsilon$, must be significant in comparison with the average instrumental background which corresponds to a total energy in the galvanometer of at least kT.

The importance of these considerations may be estimated in terms of the quantity $h\upsilon/kT$ which appears prominently in the Dirac theory of specific heats, the Planck radiation theory, and in fact throughout the quantum statistics. Since Planck's constant h is of the order of 6.6×10^{-34} joule seconds and kT is approximately 4×10^{-21} joules at room temperature, $h\upsilon/kT$ has significant magnitude for frequencies of 10^{12} cps or more. This is above the frequencies with which this study is concerned. In view of these considerations it appears that a classical approach to the problem will be adequate in the present state of the art.

8.4 Spurious Voltages in the Electrical Circuit

In the construction and installation of a high sensitivity seismometer-galvanometer system one naturally attempts to select materials for the electrical conductors and to arrange the conductors so as to minimize thermo-electric voltages and stray electrical pickup in the conducting loop. In principle these spurious voltages may be reduced to zero. In practice one may be more or less successful in reducing these disturbances to insignificant levels. In this paragraph we shall show that one may readily compute the quantitative effect of such voltages as a guide to indicate their relative importance in any given system.

Let us take as an example a direct coupled system having negligible inductance, as shown in Fig. 8.1. Here e_m represents the combined effects of thermo-electric junctions under the influence of fluctuating temperature and the emf due to magnetic induction. e_s is taken as the voltage generator simulating electro-static induction in series with c, the capacity between conductors in the inter-instrument cabling. It has been assumed, with respect to e_s that all the resistance is at the seismometer. This is the worst case with respect to electrostatic induction.

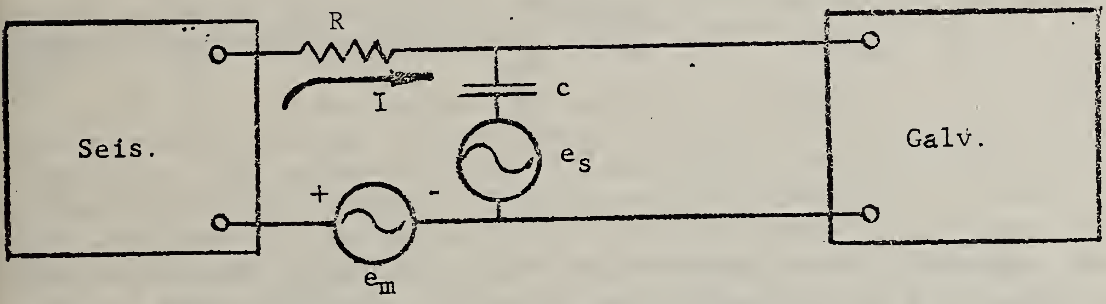

Figure 8.1

The pertinent equations may be written

$$-M\frac{d\dot{y}}{dt} = \left(M\frac{d^2}{dt^2} + S\right)x - G_1 I \qquad 8.4.1$$

$$0 = \left(K\frac{d^2}{dt^2} + U\right)\theta - G_3\left(I + c\frac{de_s}{dt}\right) \qquad 8.4.2$$

$$e_m = G_1\frac{dx}{dt} + G_3\frac{d\theta}{dt} + RI \qquad 8.4.3$$

Eliminating x and I, letting $d/dt = j\omega$, remembering that $2\lambda_o\omega_o = G_1{}^2/RM$ and $2\lambda_g\omega_g = G_3{}^2/RK$ one may write

$$\sqrt{2\lambda_o\omega_o} \quad \sqrt{M}\,\dot{y} + \left(\frac{\omega_o^2 - \omega^2}{\omega^2}\right)\frac{e_m}{\sqrt{R}} + \left(\frac{\omega_o{}^2 - \omega^2 + 2j\,\lambda_o\omega_o\omega}{\omega^2}\right) j\omega c\sqrt{R}\;e_s =$$

$$\left\{\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)\left(\frac{\omega_g}{\omega} - \frac{\omega}{\omega_g}\right) + 2j\,\lambda_o\left(\frac{\omega_g}{\omega} - \frac{\omega}{\omega_g}\right) + 2j\,\lambda_g\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)\right\}\frac{\omega_o}{\sqrt{2\lambda_g\omega_g}}\;e\sqrt{}$$

$$8.4.4$$

The reader will note that each of the four terms in equation 8.4.4 has the dimensions of the square root of power. More important is the fact that a given earth velocity multipled by the coefficient of $\dot{y}$ and divided by the coefficient of either e_m or e_s will give the magnitude of that foreign voltage which, at that frequency, will produce a galvanometer deflection equal in magnitude and phase to that produced by the given earth velocity. The galvanometer parameters do not enter the ratio of earth velocity to either foreign voltage for a given galvanometer excursion

In the equations for the various direct coupled cases, the reader may have noticed that only the ratios of circuit resistance to the critical damping resistances entered the final equations. No preference between high and low impedances could be inferred from the equations. Here we see that the effect of thermoelectric or electromagnetically induced voltages is reduced by raising the instrument critical damping resistances as high as engineering limitations will permit so as to have the highest possible circuit resistance. The improvement will be proportional to the square root of the increase of resistance. On the other hand, the effect of electrostatic induction is reduced by lowering the critical damping resistances and circuit resistance. A practical compromise would minimize the combined effect.

It is probable that the electrostatic effects can be reduced to negligible proportions by reasonable shielding of the instruments and cable as well as the use of twisted pair cable. If any added resistor or notch filter galvanometer is used, it should be placed at the working galvanometer rather than at the seismometer. Certainly all foreign voltages will be reduced by keeping the cable lengths between instruments as short as possible and enclosing the instruments in conducting containers.

It may be very difficult to eliminate the effects of electromagnetic induction in the coils of the instruments and thermoelectric emfs at cable junctions, binding posts, etc.

As an aid in visualizing the frequency dependence of the effect of thermoelectric voltage, and to show the effect of interchanging natural frequencies, the values in Table 8.1 have been computed. We have chosen a fourth-order-flat-velocity direct-coupled system, with natural frequencies of 0.8 cps and 5.0 cps and a series loop resistance of 144 ohms. From equation 6.3.6, λ_o is 0.512. The seismic mass is one kilogram. The values of e_m are those equivalent to earth displacements of 1 millimicron (10^{-9} meters) so that a number having larger absolute magnitude represents a more desirable situation.

Equation 8.4.4 shows that at the natural frequency of the seismometer a thermoelectric voltage has no effect on the record and Table 8.1 shows a very sharp maximum in equivalent thermoelectric voltage at this frequency.

In the development of specific amplitude response equations it has been pointed out that overall system gain is quantitatively independent of whether the seismometer or galvanometer has the lower natural frequency. With regard to the effect of thermoelectric voltage it can be seen that the system in which the seismometer has the lower natural frequency is slightly more tolerant in the pass-band, much more tolerant at frequencies below the pass-band of interest and slightly less tolerant at frequencies above the pass-band, to foreign voltage in the electrical circuit.

8.5 Air Buoyancy

The vertical accelerations of the earth's surface and the pier as a result of changes of atmospheric pressure are not considered as limitations on system sensitivity since purely seismic equipment cannot distinguish among various causes of pier motion.

The buoyant effects of the atmosphere acting directly on a seismic system will depend in large measure on the details of construction of individual instruments. In horizontal instruments bouyant effects, since they are perpendicular to the axis of motion, probably may be neglected.

Vertical component instruments, on the other hand, are sensitive to variations of atmospheric pressure. Diurnal variations of atmospheric pressure amounting to several percent are the rule in the temperate zones and, insofar as they are outside the pass-band of the equipment, the effect is chiefly to vary the rest-point of the inertial mass. This presents no problem for an instrument with an adequate range over which the rest-point may be varied without significant change in operating parameters.

The effects of barometric pressure fluctuations within and adjacent to the instrument pass-band are more severe. For any particular seismic system they may be estimated according to the following line of reasoning.

Table 8.1

Thermal Voltage Equivalent to One Millimicron Earth Displacement

Fourth Order, flat velocity, direct coupled system, one kilogram seismic mass, natural frequencies 0.8 and 5.0 cps, resistance 144 ohms

Frequency cps	Earth Velocity millimicrons/second	Equiv. Voltage - Microvolts Seis at 0.8 cps	Seis at 5 cps
0.10	0.628	$+ 2.7 \times 10^{-4}$	$+ 1.7 \times 10^{-5}$
0.20	1.26	$+ 2.3 \times 10^{-3}$	$+ 1.4 \times 10^{-4}$
0.50	3.14	$+ 5.5 \times 10^{-2}$	$+ 2.2 \times 10^{-3}$
0.70	4.40	$+ 3.9 \times 10^{-1}$	$+ 6.0 \times 10^{-3}$
0.80	5.02	$\pm \propto$	$+ 9.0 \times 10^{-3}$
1.0	6.28	$- 4.7 \times 10^{-1}$	$+ 1.8 \times 10^{-2}$
2.0	12.6	$- 4.1 \times 10^{-1}$	$+ 1.6 \times 10^{-1}$
5.0	31.4	$- 8.8 \times 10^{-1}$	$\pm \propto$
10	62.8	- 1.7	- 5.7
20	126	- 3.4	- 9.1
50	314	- 8.6	-21.6

The equations of motion of a seismometer all involve a term $(X - Md^2y/dt^2)$, c.f. equation 2.6.10, from which frequency response equations have been developed by suitable combinations with other relations appropriate to the system under consideration. In obtaining equations for response to earth motion, y, we have assumed in the course of the algebraic manipulation that there were no forces acting directly on the inertial mass, i.e. X was taken as zero. Here we are concerned with the system response to forces applied directly to the inertial mass when earth motions are presumed zero. Had we made the alternative assumption that $y = 0$ and performed precisely the same algebraic manipulations with X as the independent variable we would have the previously obtained frequency response equations with y replaced by X/ω^2M or $\dot{y}$ replaced by $-jX/\omega M$. (Since we are concerned with absolute amplitude relations the coefficient $-j$ may be ignored). Therefore, the galvanometer response to sinusoidal earth motion and to bob force will be the same when

$$X/\omega^2M \sim y \quad \text{or} \quad X/\omega M \sim \dot{y} \qquad 8.5.1$$

Let $\Delta\rho_a$ be the change in density of air and ρ_M be the effective density of the bob. Then the variations of buoyant force $X = gM\Delta\rho_a/\rho_M$, and equations 8.5.1 become

$$\frac{g\Delta\rho_a}{\omega^2\rho_M} \sim y \quad \text{or} \quad \frac{g}{\omega}\,\frac{\Delta\rho_a}{\rho_M} \sim \dot{y} \qquad 8.5.2$$

The effect of air buoyancy changes is, therefore, independent of the mass of the bob and becomes important as the frequency is lowered. Also, obviously, the effective density of the bob should be high.[3]

As seismometers are usually used, the variations of air density with humidity and temperature at constant pressure occur so slowly as to be lumped with diurnal pressure changes. Variations of air density with pressure are given by $\Delta P/P = \Delta\rho_a/\rho_a$ with sufficient accuracy for our purposes. Equations 8.5.2 may, therefore, be written

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[3] This might lead to a preference for brass over aluminum for the bob, but would not justify the use of lead.

$$\frac{g\rho_a \Delta P}{\rho_M P\omega^2} = y \quad \text{or} \quad \frac{g\rho_a \Delta P}{\rho_M P\omega} = \dot{y} \qquad 8.5.3$$

There seems to be very little data in the literature on variations of atmospheric pressure to be expected at periods on the order of one to thirty seconds. To get some idea of the order of magnitude of the effects to be considered one may assume an inertial mass having an effective density of 8.6 g /cc (brass), the density of air to be 1.2 mg/cc and compute the magnitude of that pressure change which is equivalent to one millimicron (10^{-9} meter) of earth displacement at a period of 6 seconds ($\omega = 1$). The computed pressure change is 0.5 microns of mercury or 0.7 dyne/cm^2. This is the change in buoyancy which would result, in a quiet atmosphere, from raising the instrument 8 mm!

From the above it seems obvious that any vertical seismometer intended for use in a high gain system should be sealed in a rigid case.

8.6 Air Convection

The possibility that convection currents around the inertial mass result in spurious deflections of the galvanometer should be considered. This again represents a possible limit in practice rather than in principle.

Some data bearing on this question was presented by Messrs. H. A. Bowman, L. B. Macurdy and H. E. Almer at the meeting of the American Association for the Advancement of Science held in Washington, D. C. in December 1958.[4]

They found that with the air at the top of a balance case 0.1°C colder than that at the bottom, convection currents were set up which produced erratic motions of the balance pointer equivalent to as much as several tenths mg unbalance of the pans (peak to peak). The pans of the balance used each had an area of 84 square cm. If one assumes that convection currents of comparable magnitude exist around a seismic mass of comparable projected area then Equation 8.5.1 may be applied. Assuming 0.1 mg unbalance at a period of 6 seconds ($\omega = 1$) and a mass of one kg the effect would be equivalent to an earth motion of 0.98 microns or 0.98 microns/sec. Turbulence due to vertical thermal gradients will, of course, be absent when the gradient is less than about

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[4] To be published in NBS Jour. of Research.

0.01°C per meter (the adiabatic lapse rate for dry air) so that in practice there is a naturally occurring stabilizing effect against turbulence due to vertical gradients.

Horizontal gradients produce similar effects without, however, a stabilizing effect analogous to the adiabatic lapse rate. Fortunately when the vertical gradient is less than the adiabatic lapse rate it serves to suppress the effect of horizontal gradients, especially when the latter are small.

These gradients are, however, clearly factors to be reckoned with. Three courses appear open to the designer. 1) The sealed, rigid case suggested in Section 8.4 may be made of a good thermal conductor and it may be supplemented by thermal insulation. 2) He can provide a close fitting shield around the inertial mass and the moving portions of its support. By close, we mean clearance on vertical surfaces of the order of 3 mm and on horizontal surfaces on the order of 6 to 8 mm. 3) He can arrange to keep the top of the seismometer case slightly warmer than the bottom by dissipating a watt or so of power at the top of the case. This is the most reliable way of avoiding convection currents.

Convection currents around a galvanometer mirror would, at first glance, appear to be more troublesome than those around the inertial mass. Fortunately several factors generally existing in a seismic system tend to suppress this effect. First the system response to earth motion is effectively increased in magnitude at this point and second, the galvanometer case is generally small. This decrease in size makes it inherently more nearly isothermal and also the suspension is automatically moderately well baffled. The use of a light beam does imply the presence of a local "hot spot" wherever light is absorbed in either the suspension or the case, so that care should be taken to be sure that the beam falls only on reflecting portions of the mirror and is of as low a total intensity as can be used (this will also tend to reduce thermo-electric effects in the galvanometer).

8.7 Light Pressure and the "Radiometer Effect"

The possibility that photon pressures could result in spurious signals which are large enough to concern a seismologist seems utterly ridiculous until one realizes that there is no a priori reason to assume that their effects are either greater or less than Brownian motion effects.

A seismometer in a sealed opaque case, all of whose parts are at the same temperature, is in a region of uniform radiation density and the inertial mass is therefore subjected to no unbalanced forces due to light pressure. On the other hand a galvanometer with an optical sensing system operates under quite different conditions of photon balance. A general solution for the forces involved is not so graphic as a numerical example.

Assume that the total photon emission of the galvanometer lamp is one watt and that the collimating lens is 2 cm in diameter at a distance of 10 cm from the filament. The lens subtends a solid angle at the filament of 0.01 π steradian or 2.5×10^{-3} part of the sphere. Assuming 60% losses in the optical system the energy striking the mirror is one milliwatt or 10^4 erg/sec. Assuming that the center of the light spot is 1 mm off the axis of rotation of the galvanometer and that the light is totally reflected then the torque due to light pressure is

$$2 \times 10^4 \times \frac{1}{3 \times 10^{10}} \times 10^{-1} = \frac{2}{3} \times 10^{-7} \text{ dyne cm.}$$

If the stiffness of the galvanometer suspension is 0.12 dyne cm/radian the resulting deflection is 5.56×10^{-7} radian.

The energy stored in the galvanometer suspension is

$$\frac{1}{2} (U\theta) \theta = \frac{1}{2} \times \frac{2}{3} \times 10^{-7} \times 5.56 \times 10^{-7} = 1.85 \times 10^{-14} \text{ ergs.}$$

This may be compared with the Brownian motion energy for one degree of freedom which, at room temperature, is about 4×10^{-14} erg (averaging 1/2 kinetic and 1/2 potential). That is, for the assumed conditions, the steady state deflection due to photon pressure represents about the rms value of the deflection due to Brownian motion, and therefore photon pressure per se need give us no concern.[5]

The radiometer effect on a galvanometer coil is not capable of so clean cut an evaluation as photon pressure. Since early attempts to detect photon pressure were completely obscured by the radiometer effect, there is a strong suspicion that it is several orders of magnitude larger than photon pressure. Certainly, in any case, a light beam should be well centered on the galvanometer mirror and so masked that it cannot strike less reflecting parts of the galvanometer.

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[5] For instance, a 10% change of light intensity would represent one-tenth the effect of Brownian motion. It would be difficult to balance a phototube amplifier so that it would tolerate so large a change in lamp intensity. Therefore, the effect of variation of light intensity on the phototube would swamp the variation of light pressure on the galvanometer mirror.

8.8 Magnetic Noise

The operation of any transducer whose output depends upon the interaction of a magnetic field and a conductor will be adversely affected by any effects which cause the field to change in an unpredictable manner. One such effect, known as the "Barkhausen effect" or "Barkhausen noise", results in abrupt changes or steps in the magnetic field associated with ferromagnetic materials. Its origin is presumed to be abrupt upsets of individual magnetic domains. Equipment capable of demonstrating Barkhausen noise is easily assembled and it is readily observed when the magnetic induction of a ferromagnetic material is sufficiently great that increased induction is not readily obtained by the orderly growth of favorably oriented domains at the expense of their neighbors. This form of noise is only observed as the magnetic induction is changed and is not detectable at low values of the induction.

In moving coil transducers the magnetic induction of the ferromagnetic materials is perturbed only by the currents generated in the electrical circuits. The perturbations may safely be considered as too small to generate Barkhausen noise.

In transducers of the variable reluctance type there may be relatively large changes in magnetic induction of the iron within the coils as the mass moves relative to the frame, so that it may in fact be possible for the Barkhausen effect to be present and generate spurious signals. In instruments of the variable reluctance type, as presently constructed, Barkhausen noise is avoided by keeping the total magnetic induction of the iron sufficiently low.[6]

Another source of magnetic field fluctuation must exist in principle, although to the authors' knowledge it has neither been named nor observed. Consider a magnetic structure composed of a permanent magnet, soft iron and an air gap. Since the atoms of both the magnet and the iron are in motion due to thermal agitation, there must be random fluctuations in both the magnetomotive force and the circuit reluctance. These fluctuations result in random fluctuations of the strength of the field in the air gap. Since the magnet is presumed to be in thermal equilibrium with its surroundings and with any electrical circuit to which it may be coupled, again we have a circumstance wherein the act of coupling may change the spectral distributions of the energy fluctuations although it cannot alter the time average of the thermal energy in either the electrical or magnetic circuit (cf. subsection 8.2). We have therefore another instance of a noise source whose effect on the system is not to reduce its theoretical ultimate sensitivity.

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[6] A 100 kg horizontal instrument of the Benioff variable reluctance type was examined for Barkhausen noise by Mr. Sanford of the Magnetic Measurements Section. The effect was not detected.

By way of recapitulation we should like to remark that thermal agitation as manifest by fluctuations of a magnetic field is nameless. When it is observed as the motion of a particle it is called "Brownian motion". When observed as a voltage fluctuation, the phenomenon is called "Johnson noise". The use of different names for different manifestations of the same basic cause does not warrant the assumption that by coupling between the phenomena one can cause thermal energy to flow persistently from one system to another at the same temperature, in violation of the second law of thermodynamics.

8.9 External Magnetic Fields

The possibility that a high gain seismometer using an electromagnetic transducer may be influenced by externally created weak magnetic fields must be considered. The magnitudes of these effects will be markedly a function of the detailed construction of individual instruments so that we can only indicate some of the coupling mechanisms involved, leaving calculations or measurement of the expected magnitudes to the designer or user. A magnetic field, whether uniform or having a gradient in space, which is constant in time will affect -- at most -- the rest-point of the instrument. When a magnetic field varies in time, a number of possible effects must be considered.

8.9.1 Field Uniform in Space, but Varying in Time

Although it is possible to have a pier and shelter in which the magnetic field is quite uniform before installation of an instrument the presence of magnetic materials in the instrument will introduce gradients of the field so that the uniform field condition is a somewhat idealized situation. For a uniform field the effect of time variations is chiefly to introduce spurious voltages in the coils of the instrument (cf. Section 8.4). One may assume as a worst case that the iron surrounding a moving coil instrument presents no shielding or as a best case that the iron contained in the coils of a variable reluctance pick-up has a negligible perturbing effect.

It is obvious, though not generally realized, that a magnetic field which would be uniform in space in the absence of a magnetic material can exert only torques on the material. Such a field cannot exert a translational force even though the magnetic object has fringing fields or definite poles of its own. In considering a specific instrument it must be recognized that the structural linkages may couple this torque into the degree of freedom being monitored. These points become of special interest when one considers springing a magnetic structure as its own seismic mass in an attempt to build an instrument having a high ratio of sprung mass to total mass.

8.9.2 Field Non-Uniform in Both Space and Time

When a magnetic field does have a space gradient then translational forces are exerted on magnetic material and the time variation of these forces must be considered. These effects are in addition to the spurious voltage mentioned above. They could well be quite significant during geomagnetic storms especially on a sprung mass having fringing fields.

For any given instrument these effects can be calculated (at least in principle) and the computed forces inserted in equations 8.5.1; however, the user will certainly prefer the result of tests on assembled instruments.

Section 9

CALIBRATION

9.1 General

In considering the calibration of a seismic system it will be convenient to discuss the subject under three headings: 1) Over-all system calibration by use of a shake table, 2) application of known forces to the inertial mass or known emfs in the electrical circuit (semi-direct calibration), 3) measurement of appropriate parameters of individual components or sub-assemblies, (indirect calibration). This choice is admittedly somewhat arbitrary.

9.2 Shake Table

To the user who is interested in knowing the performance characteristics of his equipment with an accuracy of say 1 to 5%, a shake table which simulates pier motions represents the most direct way of obtaining over-all system calibrations. Unfortunately a good shake table, aside from its cost, is a piece of laboratory equipment, and laboratories are usually noisy locations. Thus calibrations must be run against the relatively high noise level of the laboratory. This requires that the seismometer have a large dynamic range, so that the parameters at high and low levels are the same. Also, there is always the possibility that after calibration, the instrument parameters may change during shipment and installation.

Many workers have attempted to devise relatively cheap and simple shake-tables for permanent installation between the pier and the instrument. As such a device seems not to have been generally adopted, we may presume that the attempts have not been too successful.

9.3 Semi-direct Calibration

In the course of developing equations relating a recordable output to pier motions, relations were noted between the pier motion and forces applied directly to the inertial mass which will produce equal recorded output (cf. equations 8.5.1). Also, a relation has been shown between spurious voltage in the electrical circuit and earth motion for equal recorded output (cf. equation 8.4.4). For want of a better term we will call methods based on these relations semi-direct to imply that they represent calibration of an over-all, operational system, but are not so unambiguous as the shake table, since one must presume that a relation such as equation 8.5.1 is valid.

In this section we will discuss possible ways of calibrating an operational system based on these techniques.

9.3.1 Weight Lift

The weight lift measurement has become a well recognized way of estimating over-all system performance and "magnification". The test is easily performed, at least on unsealed instruments. The recorded output is that resulting from a step-function in the force applied to the bob. If the mass of the bob is known, this disturbance is equivalent to a step in the acceleration of the pier. If the effective stiffness of the spring is known, the disturbance is equivalent to the release of the bob from a non-equilibrium position. From the detailed shape of the record the user can, in principle, compute a relative amplitude-frequency response curve. In practice one can only infer that the system performance has or has not changed by comparison with previous records. The peak amplitude on the record, resulting from the weight lift, is related to the sinusoidal midband gain of the system; however, in the present state of the art one should use an arbitrary constant determined as the result of shake-table studies. The use of such an arbitrary constant presupposes that the frequency response of the system remains unchanged.

With all its limitations the weight lift has the advantages of simplicity and unambiguous polarity indication. Unfortunately the engineering required to arrange a weight lift calibrator within a sealed unit takes it out of the "simple" class.

9.3.2 Pneumatic Calibrator

In subsection 8.5, it was pointed out that the vertical component seismometer is quite sensitive to small variations of atmospheric pressure so that a high gain instrument should be enclosed in a rigid, sealed container.

If we are committed to a sealed, rigid container for the instrument it is logical to examine the possibility of using buoyancy forces to calibrate the system.

Referring to subsection 8.5, a calibrating signal equivalent to 50 millimicrons peak to peak at 6 sec period requires a density change on the order of 35×10^{-6}, if the inertial mass has an effective density of 8.6 gm/cc and the case is filled with gas to a density of .0012 gm/cc (air at 1 atmosphere pressure and room temperature).

Suppose the case has a volume of 10 liters and is connected by fine tubing to a 1 cc hypodermic syringe. The required volume change of 0.35 cc peak to peak may be produced by a motor driven eccentric with a stroke of the order of 2 cm. With a variable speed motor one may cover a range of frequencies and remote operation is quite feasible. Using equation 8.5.3 one may obtain the operating amplitude-frequency response curve of the over-all system.

This idea has at least two disadvantages. 1) It cannot be directly applied to horizontal instruments. 2) As outlined above, inadvertent polarity reversals within the system are not easily detected. (A second syringe with solenoid withdrawal and spring return will indicate polarity).

9.3.3 Electromechanical Calibrator

Equations 8.5.1 suggest that any mechanism which will apply known sinusoidal forces to the inertial mass may be used to obtain an amplitude-frequency response curve for the over-all system.

Such a device might consist of a motor-driven eccentric coupled to the mass by a light spring. From the measured spring stiffness (a watch or clock hairspring would probably be about right) and the eccentricity of the drive, the applied force is easily obtained. Again a variable speed motor is indicated. Among the engineering difficulties to be considered are gear noise and electrical cross-talk between the motor and the main coils of the seismometer since the whole mechanism would certainly be mounted within the case.

This device, like the remotely operated sinusoidal pneumatic calibrator, cannot check a system for polarity.

9.3.4 Electromagnetic Calibrator

Another method for subjecting the inertial mass to known sinusoidal forces is to provide the seismometer with a second coil working in an auxiliary gap preferably arranged to have minimum electromagnetic coupling to the main coil and its circuit.

Calculation of the motor constant required to have one milliampere equivalent to 50 millimicrons indicates that a few meters of wire working in a field on the order of 10 to 100 gauss is all that is required for a few kilograms at 2 cps.

Such an arrangement is as readily adapted to remote operation as a pneumatic or electromechanical calibrator and has several advantages over either, provided line losses can be measured and presumed constant. The amplitude and frequency of the input signal is readily altered over a wide range. The device is not even restricted to sinusoidal inputs, a step function of calibrator current accurately simulates the weight lift. One may determine the system response to any transient pier motion for which, using equation 8.5.1, a suitable time varying current can be generated. Obviously the calibrator must be driven from a constant current generator, i.e. it must see a source impedance high enough so that the emf's generated in the calibrator coil by bob motion produce negligible modulation of the calibrator current. This presents no special engineering difficulties.

Of the many possible methods for determining the motor constant of the calibrator coil several will be discussed in subsection 9.4. One method is so direct and specifically applicable to an operational seismic system as to be properly included here. It consists of matching the record produced from a weight lift by the output due to a step-function of current. The weight lifted should produce a record comfortably above the local background level but not so great as to approach non-linearities in the system. Here the system is used only to compare two forces and is only required to be stable throughout the course of the measurements. Having obtained the motor constant of the calibrator one may then energize the calibrator sinusoidally to obtain the amplitude-frequency response curve for the over-all system.

Over-all system polarity may be checked by verifying the polarity of the circuits used to energize the calibrator. This is generally simpler than checking the polarities of the several instruments and connections in the signal circuits of the system.

9.3.5 Electrodynamic Calibrator

The two engineering problems which exist when using an electromagnetic calibrator: namely, "cross-talk" between the calibrator and recording circuits, and the possible ambiguity of polarity, have led Mr. Ben S. Melton to point out that an electrodynamic calibrator may be devised which replaces these problems with others which may be more tolerable.

If one arranges two coils so that their magnetic fields interact and passes currents through them then a force will be developed between the coils which is proportional to the product of the currents through them. In particular, for two coils series connected, the instantaneous force is proportional to the square of the current. Thus, the polarity of the force is independent of the polarity of the current and, for sinusoidal excitation the resultant force also varies sinusoidally, but at double frequency.

Attaching one coil to the instrument frame, the second to the mass, and connecting them in series gives a calibrator whose polarity is determined by the construction. Further, the output from the seismic system should be separable on the record from system "cross-talk" since the former is at double the driving frequency.

This technique reduces the problems of polarity ambiguity and "cross-talk" inherent in electromagnetic calibrators at the expense of being able to calibrate with signals displacing the mass to only one side of the operating rest point. In order to inject signals of arbitrary amplitude and wave-form, one must generate a current in which the instantaneous amplitude is the square root of the desired instantaneous pier acceleration.

The same calibration arrangement can be excited by an amplitude modulated carrier of frequency comfortably above the pass band of the system. The attenuation of a transmission line may be larger and more variable for a carrier signal than for one of the modulation frequencies.

Many users would prefer the limitations of an electrodynamic calibrator to those of an electromagnetic device.

9.3.6 Injected Voltage

9.3.6.1 Amplitude of Series Injected Voltage

The effect of spurious voltages in the electrical circuit between the seismometer and galvanometer has been discussed in subsection 8.4. It should be possible to turn this sensitivity to advantage and use it as a method of calibration. To cover a range of frequencies (cf. Table 8.1) it appears desirable to include several four-lead precision resistors in series as a portion of the total loop resistance. Known currents through one of these resistors can be related to equivalent earth motion by an equation such as equation 8.4.4.

This method appears to have two advantages over those considered so far. First, one may make an overall system calibration without the need for modification of the seismometer to include an electro-mechanical or electromagnetic calibrator. Second, it appears serviceable as a means of calibrating a system which consists of several matched seismometers, separated in space, series or parallel connected to a single galvanometer.

The problem of polarity determination could be quite severe, especially if "the seismometer" is a group of interconnected "matched" instruments.

9.3.6.2 Bridge Methods

The authors have not attempted to develop methods for calibrating assembled seismic systems utilizing bridge measurements. One such possibility is that of Willmore which appears to be useful for calibrating installed systems having instruments with readily lockable masses.

9.3.6.3 Phase Measurements

It is of interest to consider phase measurements using the circuit of Fig 4.2 or Fig 4.3. For example, one may open the electrical circuit of a direct coupled system and consider it as a two terminal network. Neglecting mechanical losses one may write the impedance from equation 4.3.4 as

$$Z = R \left\{ 1 + 2\, j\omega \left(\frac{\lambda_o \omega_o}{\omega_o^2 - \omega^2} + \frac{\lambda_g \omega_g}{\omega_g^2 - \omega^2} \right) \right\} \qquad 9.3.1$$

By inspection it can be seen that the imaginary part approaches infinity as ω approaches ω_o or ω_g. The imaginary part is zero when

$$\lambda_o\omega_o\,(\omega_g^2 - \omega^2) + \lambda_g\omega_g\,(\omega_o^2 - \omega^2) = 0 \qquad 9.3.2$$

Equation 9.3.2 with $\omega = \omega_v$ is equivalent to the defining equation of ω_y (cf. equation 6.2.4) so that measuring these three phase points give the natural frequencies of the seismometer and galvanometer and also the ratio of their critical damping resistances.

In what follows it will be convenient to rearrange equation 6.2.4 or 9.3.2 slightly and define a quantity B as

$$\frac{\lambda_o\omega_o}{\lambda_g\omega_g} = \frac{\omega_o^2 - \omega_v^2}{\omega_v^2 - \omega_g^2} = \frac{\mathcal{R}_o\omega_o}{\mathcal{R}_g\omega_g} \equiv B^2 \qquad 9.3.3$$

Using equation 9.3.3 to eliminate λ_o and λ_g from equation 6.3.1 we may write the general fourth order direct coupled response as

$$\frac{\omega}{\omega_m}\sqrt{\frac{M}{K}}\,\frac{y}{\Theta} = \frac{\omega_m}{\omega}\left(\frac{\omega_g}{\omega_o}\,B + \frac{\omega_o}{\omega_g}\cdot\frac{1}{B}\right) - \frac{\omega}{\omega_m}\left(B + \frac{1}{B}\right)$$

$$- \frac{1}{2\lambda m}\left[\left(\frac{\omega_m}{\omega} + \frac{\omega}{\omega_m}\right)^2 - \left(\frac{\omega_g + \omega_o}{\omega_m}\right)^2\right] \qquad 9.3.4$$

If we can determine λ_m then the phase-frequency response curve and the relative amplitude-frequency response curve are known. A knowledge of the effective mass of the seismometer and the moment of inertia of the galvanometer serve to make the amplitude-frequency response curve quantitative.

Either of the following two measurements of phase, together with a measurement of the d-c resistance of the seismometer-galvanometer circuit, serve to determine λ_m.

Given a suitable inductor with resistance R_L and inductance L connected in series with the seismometer and galvanometer, the impedance of the circuit (again neglecting mechanical losses) may be written

$$Z = R_L + R\left\{1 + j\omega\left[\frac{L}{R} + 2\left(\frac{\lambda_o\omega_o}{\omega_o^2 - \omega^2} + \frac{\lambda_g\omega_g}{\omega_g^2 - \omega^2}\right)\right]\right\} \qquad 9.3.5$$

The imaginary part of equation 9.3.5 approaches infinity at $\omega = \omega_o$ and $\omega = \omega_g$ as before. It is zero when

$$\lambda_m\omega_m = \frac{L}{2R} \cdot \frac{(\omega_o^2 - \omega^2)\,(\omega_g^2 - \omega^2)}{(\omega^2 - \omega_v^2)\cdot(B + \frac{1}{B})} \qquad 9.3.6$$

Each of the two frequencies satisfying Eq. 9.3.6 will be slightly above the natural frequency of one of the two instruments.

Alternatively one may connect a suitable condenser of capacity C_e in parallel with the network terminals. The reactive component of impedance will be zero at two frequencies each of which is slightly below the natural frequency of one of the instruments. The imaginary part of the impedance of the parallel combination vanishes when

$$\lambda_m\omega_m = \frac{(\omega_g^2 - \omega^2)(\omega_o^2 - \omega^2)}{4\omega^2 RC_e(\omega_v^2 - \omega^2)(B + \frac{1}{B})}\left(1 \pm \sqrt{1 - 4\omega^2 RC_e}\right) \qquad 9.3.7$$

9.4 Indirect Methods

Under indirect methods we include all methods of calibration which involve determination of the parameters of the seismometer, galvanometer, T pad if any, and recording mechanisms as separate components since these methods require that the over-all system be presumed to respond according to a known equation such as those developed in earlier sections.

9.4.1 Recapitulation

The performance of an inertial type seismometer having negligible internal losses using a variable reluctance transducer is completely defined by a knowledge of M, ω_o, ω_g, R and L for the instrument. All these parameters except M can be measured by electrical means as indicated in paragraph 3.2.1.3. M may be measured directly if the instrument is disassembled. Alternatively it can be obtained from a measurement involving one mechanical quantity, e.g. a motor constant.

The performance of a galvanometer having negligible internal losses and inductance is completely defined by K, ω_g, R and $\mathcal{R}_g$. Measurement of these parameters has been discussed in subsection 4.4.[8]

A moving coil seismometer may also be defined by the methods discussed in subsection 4.4. Equation 4.4.1 will require either a micrometer microscope, an electrical contact mounted on a micrometer screw or a similar method of determining mass position.

9.4.2 Special Tests

In connection with checking the "linear range" of a moving coil instrument several tests have been used for measuring the motor constant directly, for both the main coil and an auxiliary electromagnetic calibrator.

Adjustment of the "rest point" of an instrument is readily made, for test purposes, by injecting a constant d-c current from a high impedance source into the main coil of the instrument.

A specific mass position may be sensed by means of an electrical contact (it must be very light and on a very compliant spring) which can be precisely adjusted in the line of mass motion.

The motor constant of an instrument can be obtained by a force balance using the electrical contact to determine the balance point. For instance (testing a vertical instrument) one may bring the mass to the desired restpoint, adjust the electrical contact to the "just make" position, add a known mass to the inertial mass and then by changing the d-c current in the coil return the mass to its initial position as indicated by the electrical contact. The force applied by the known mass, divided by the change of d-c current, gives directly the motor constant at the initial rest-point.

Using a ballistic galvanometer calibrated in volt seconds and providing the inertial mass with stops allowing only a small, precisely known, travel, one may easily determine an average generator constant from the galvanometer deflections obtained on moving the inertial mass abruptly from one stop to the other. Parenthetically it may be noted that the generator constant is usually thought of as volts per (cm per sec). This is volt sec per cm, the quantity measured directly by this test.

The following method is somewhat tedious in the taking of data and laborious in its rectification but, knowing the effective mass, it is capable of approaching 1% accuracy in the simultaneous determination of the motor/generator constant of both the main and calibrator coils of an instrument having negligible inductance.

Consider the circuit of Fig 9.1. Here we may assume

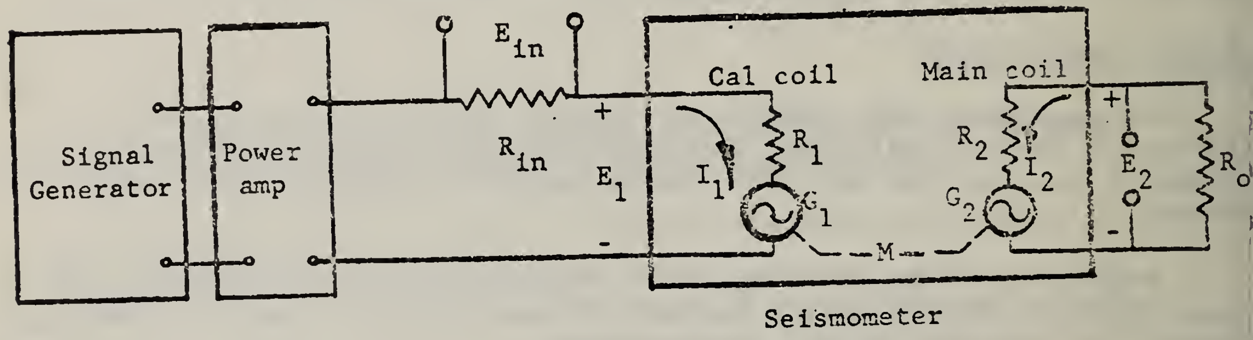


Fig 9.1

$X = y = x_o = L_1 = L_2 = 0$ and in a well designed instrument the electromagnetic coupling between the coils will be negligible so that $M_{12} \sim 0$.

Equations 2.8.6 and 2.8.8 may be written

$$0 = \left(\frac{d^2}{dt^2} + \frac{r_x}{M}\frac{d}{dt} + \omega_o^2\right) x - \frac{G_1 I_1}{M} - \frac{G_2 I_2}{M} \qquad 9.4.1$$

$$0 = G_2 \frac{dx}{dt} + I_2 (R_2 + R_o) \qquad 9.4.2$$

Eliminating x and substituting $E_{in} = R_{in} I_1$ gives

$$0 = \left[\frac{d^2}{dt^2} + \left(\frac{r_x}{M} + \frac{G_2^2}{M(R_o + R_2)}\right)\frac{d}{dt} + \omega_o^2\right]\left(\frac{I_2 (R_2 + R_o)}{G_2}\right)$$

$$+ \frac{G_1}{M}\frac{d}{dt}\left(\frac{E_{in}}{R_{in}}\right) \qquad 9.4.3$$

Letting $d/dt = j\omega$, remembering (from Fig 4.1) that R_x may be considered as a resistor internal to the instrument with a numerical value equal to G_2^2/r_x and that $E_2 = - I_2 R_0$ gives

$$\frac{E_{in}}{E_2} = \frac{G_2}{G_1}\frac{R_{in}}{R_x}\left(\frac{R_o + R_2 + R_x}{R_o}\right) - j\left(\frac{\omega_o^2 - \omega^2}{\omega}\right)\left(\frac{R_o + R_2}{R_o G_2}\right)\frac{R_{in} M}{G_1} \qquad 9.4.4$$

Multiplying by the complex conjugate gives

$$\left|\frac{E_{in}}{E_2}\right|^2 = \left(\frac{R_o + R_2}{R_o}\right)^2 \left(\frac{R_{in} M\omega_o}{G_1 G_2}\right)^2 \left(\frac{f_o}{f} - \frac{f}{f_o}\right)^2$$

$$+ \left(\frac{G_2}{G_1}\right)^2 \left(\frac{R_{in}}{R_o}\right)^2 \left(\frac{R_o + R_2 + R_x}{R_x}\right)^2 \qquad 9.4.5$$

Equation 9.4.5 is linear in the variable $(f_o/f - f/f_o)^2$. By measuring $|E_{in}/E_2|$ at a number of frequencies in the neighborhood of f_o and plotting $|E_{in}/E_2|^2$ as ordinate with $(f_o/f - f/f_o)^2$ as abscissa one may obtain the slope m, and intercept on the ordinate, b. Then

$$\sqrt{m}\, G_1 G_2 = R_{in} M\omega_o \left(\frac{R_o + R_2}{R_o}\right) \quad \text{and}$$

$$\sqrt{b}\, \frac{G_1}{G_2} = \frac{R_{in}}{R_o} \cdot \frac{R_o + R_2 + R_x}{R_x} \quad \text{from which it follows that}$$

$$G_1^2 = \left(\frac{R_{in}}{R_o}\right)^2 \frac{M\omega_o}{\sqrt{mb}} (R_o + R_2) \left(\frac{R_o + R_2 + R_x}{R_x}\right) \qquad 9.4.6$$

and

$$G_2^2 = \sqrt{\frac{b}{m}}\, M\omega_o (R_o + R_2) \left(\frac{R_x}{R_o + R_2 + R_x}\right) \qquad 9.4.7$$

A few remarks on instrumenting Fig. 9.1 may be of interest. We have used a low frequency (not d-c) amplifier and recorder with a high input impedance as a voltmeter. R_{in} was made up of a resistor in parallel with a precision potentiometer. At each frequency the amplifier was connected across R_o and a section of record was made. The amplifier was then connected across the potentiometer output and the potentiometer adjusted to give a record of the same amplitude. This avoided questions of the amplitude-frequency characteristic of the recording amplifier and its calibration as a voltmeter. A range of frequencies of ± 3/4 octave centered on f_o was adequate for plottin purposes. A value of R_o appropriate to 1/4 critical damping was satisfactory; higher values of R_o result in excessive magnification by the seismometer of any harmonic distortion in the signal generator or power amplifier as the test frequency approaches $f_o/2$. Lower values of R_o will require either more input current or a higher gain in the voltmeter. A d-c constant current supply across E_2 served to shift the operating rest point of the inertial mass for studying the change of G_1 and G_2 with mass position.

A slight error in selection of f_o for plotting the data results in a very slender parabola, frequencies above f_o lying on one half and below f_o lying on the other. The slope and intercept of its axis will give the desired values of m and b.

The circuit of Fig 9.2 gives an excellent check of the ratio of the average value of G_1 to G_2. Differences in their non-linearity over the range of mass motions are easily seen.

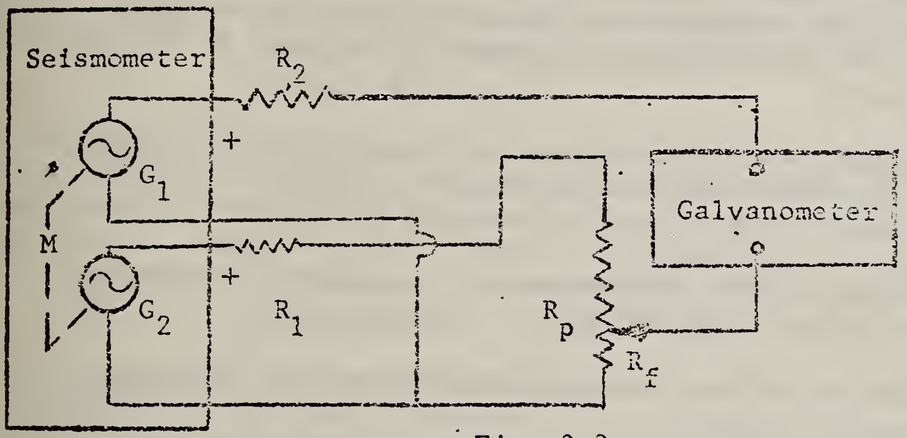


Fig. 9.2

The main coil is connected to a potentiometer having a total resistance large compared to either the coil resistance or critical damping resistance of the seismometer. At the proper adjustment of the potentiometer the galvanometer is remarkably insensitive to violent motions of the mass The ratio of G_1/G_2 is $R_f/(R_p + R_1)$. As we have used it, the galvanometer was heavily overdamped but the point seems immaterial as long as it is a sensitive instrument

Section 10

Nomenclature and Pertinent References

10.1 General

This section is in two parts. The first contains a complete list of the symbols used in this report. Each is given with a brief description of its significance and the paragraph in which it is first introduced. In several cases a single symbol has been assigned different meanings in different sections and these are noted by multiple listings. In these cases the first meaning is generally not used after the second has been introduced.

The arrangement of listings is strictly alphabetical, English preceding Greek. For each letter the arrangement is print capital, script capital, print lower case followed by script lower case. Primed symbols follow immediately after unprimed symbols. Where a symbol is used with a number of subscripts the arrangement of subscripts is numerical followed by alphabetical as indicated above.

The second part of this section contains a list, alphabetical by authors, of some of the pertinent literature. Many of the symbols used by others correspond identically to symbols or relations among symbols used in this report. Accordingly, tables have been prepared indicating some of these identities. Throughout most of this report we have presumed a seismometer in which the principal inertial mass is constrained to linear motion so that the instrument natural frequency is definable in terms of a stiffness and a mass while many authors have considered specific instruments in which the inertial mass is pivoted on an arm so that it is constrained to rotate. In this case the natural frequency is defined by a moment of inertia and a restoring torque. The similarity of concept between mass and moment of inertia and between stiffness and restoring torque has led us to include listing of the corresponding symbols with the notation that they are analogous rather than identical. The remark "analogous" has also been used for other cases of similar but not identical symbols.

10.2 Nomenclature

A	(6.3.2)	a constant introduced for simplification of algebraic manipulation. $A\omega_m \equiv 2\ (\lambda_o\omega_g + \lambda_g\omega_o)$, hence a measure of damping.
A_1	(2.1.1)	first coefficient of a coordinate of a constraint, its magnitude is determined by the linkages among the constraints.
A_m	(2.3)	area of the magnet in the instrument.
a	(2.4.1)	a weighting factor relating the axial motion of an element of mass linked to the bob and the coordinate of bob motion.
	(6.3.1)	a constant introduced to simplify algebraic notation, a measure of the octave separation of the instrument natural frequencies. $a \equiv \lvert\omega_g - \omega_o\rvert/\omega_m$
B	(6.3.2)	a constant introduced for simplification of algebraic manipulation. $B\omega_m \equiv 2\ (\lambda_o\omega_o + \lambda_g\omega_g)$
	(9.3.5.3)	a constant introduced to simplify algebraic manipulation $B^2 \equiv \dfrac{\omega_o^2 - \omega_v^2}{\omega_v^2 - \omega_g^2}$
$\left.\begin{matrix} B_1 \\ B_2 \\ B_3 \end{matrix}\right\}$	(3.2.2)	three arbitrary constants in the transient solution for a seismometer connected to a resistive load.
B_m	(2.3)	the magnetic induction in an element of a permanent magnet.
$\mathcal{B}$	(2.1.2)	the magnetic induction
b	(2.4.1)	a weighting factor relating the transverse motion of an element of mass linked to the bob and the coordinate of bob motion. Only used in this sense in this section.

b	(7.4.1)	a constant introduced to simplify algebraic manipulation. $b = (\omega_1 + \omega_2)/\omega_m$, a measure of the octave separation of the final RC filters.
	(9.4.2)	intercept on the ordinate, of a straight line obtained by rectification of data taken during measurements on a seismometer.
C_1	(7.4.1)	condenser of the first of two final RC filters. It is associated with R_1 to give a time constant τ_1 and a 3 db point at ω_1.
C_2	(7.4.1)	condenser of the second of two final RC filters. It is associated with R_2 to give a time constant τ_2 and a 3 db point of ω_2.
C_e	(4.3.2)	a capacitor added to a galvanometer, measurements on the combination permit determination of $\mathcal{K}_g$.
C_x	(3.2.4)	numerically M/G_1^2. The electrical capacity equivalent to the inertia of the bob in the electrical analog of the seismometer.
C_θ	(4.3.2)	numerically K/G_3^2. The electrical capacity equivalent to the inertia of the armature in the electrical analog of the galvanometer.
c	(7.4.1)	a symbol introduced to simplify algebraic manipulation. $c = \left\| k^2\omega_2 M\dot{y}^2/\omega_m^2\omega_1 K E_o^2 \right\|$.
	(8.3)	the capacity between the conductors connecting a seismometer and galvanometer.
D	(3.2)	defined as $d/d\tau = d/d\omega_c t$. D is the derivative operator in normalized time.
$\mathcal{D}$	(2.1.2)	the electric displacement of an electrostatic field.

dm	(2.4)	an infinitesimal element of mass of the instrument.
E	(3.2)	voltage across a seismometer and external resistor in series with it.
E_o	(2.3)	a supplementary emf necessary to preserve constant current in the coil simulating the instrument magnet.
	(4.4.1)	output of a device for producing a signal proportional to galvanometer deflection.
	(7.4.1)	output voltage of a system consisting of seismometer, galvanometer, telemetering device and final filter.
E_1	(2.2.3)	the generalized force of the electrical coordinate, the electromotive force. Also used as the external emf applied to a coil of the seismometer.
E_2	(2.8.3)	emf applied to a second real coil. A prime is used to distinguish the numerical value from that obtained under different test conditions.
E_3	(4.2)	the potential difference at the terminals of the galvanometer circuit.
E_{in}	(9.4.2)	emf across R_{in}, to monitor the driving current into a calibrator coil.
$\mathcal{E}$	(2.1.2)	Electric field strength of an electrostatic field.
e	(3.2.4)	defined as Gdx/dt.
	(7.2)	voltage across a circuit, or portion of a circuit.
e_o	(4.4.1)	output voltage of a device for converting galvanometer coil rotation into voltage during special tests.
e_1	(7.4.1)	the output of a device for producing a voltage proportional to the deflection of the galvanometer mirror.
e_2	(7.4.1)	output of a data transmission system. Input to the final pair of RC filters.
e_3	(4.4.1)	external emf applied to a galvanometer coil. (special test)

e_m	(8.3)	an extraneous voltage generated in the electrical circuit between a seismometer and galvanometer as by electromagnetic induction or thermoelectric junctions.
e_s	(8.3)	an extraneous voltage generated in the electrical circuit between a seismometer and galvanometer as by electrostatic induction.
F	(8.2)	some function of the frequency variable ω.
f	(3.2)	normalized frequency defined as ω/ω_o.
f_g	(5.2)	natural frequency of the galvanometer when the natural frequency of the seismometer is taken as unity. $f_g \equiv \omega_g/\omega_o$.
f_s	(3.2.1.1)	defined as $\sqrt{\gamma^2 + 1}$. f_s may be considered as the normalized apparent natural frequency of the instrument with a superconducting coil short-circuited at its output. It is also the series resonant frequency (normalized) of the electrical inductance and the reflected impedance of the spring-mass system.
G	(2.6)	The generator/motor constant of an instrument, i.e. emf per unit velocity or mechanical force per unit current.
G_1	(2.8.2)	the generator/motor constant associated with one real coil of the seismometer.
G_2	(2.8.2)	the generator/motor constant associated with a second real coil of the seismometer.
G_3	(4.2)	the motor/generator constant of the galvanometer.
G_N	(7.2)	when a seismometer drives two galvanometers simultaneously, G_N refers to the generator/motor constant of that galvanometer whose output is not being recorded.

G_w	(7.2)	when a seismometer drives two galvanometers simultaneously G_w refers to the generator/motor constant of that galvanometer whose output is being recorded.
g	(2.4.3)	the steady upward acceleration to which an instrument must be presumed to be subjected in order that its parts shall exert downward forces equivalent to those due to gravitational attraction.
H	(2.1.2)	the intensity of the magnetic field.
H_m	(2.3)	Magnetizing force in the magnet.
h	(8.3)	Planck's constant. $6.6\text{x}10^{-34}$ joule sec.
I	(3.2)	electrical current flowing into the seismometer from an external circuit.
I_o	(2.2.4)	the phantom current in a simulated coil replacing the permanent magnet in the instrument.
I_1	(2.2.3)	electrical current--the generalized velocity of the electrical coordinate. Also used as the current flowing into the coil of the instrument.
	(2.8.2)	current in the first real coil.
	(2.8.3)	A prime is used to distinguish the numerical value from that obtained under different test conditions.
I_2	(2.8.2)	current in a second real coil of the instrument.
I_3	(4.2)	current in the galvanometer coil.
I_N	(7.2)	when a seismometer drives two galvanometers simultaneously it is necessary to distinguish between them. I_N refers to the current through the galvanometer whose output is not being recorded.
I_{exp}	(3.2.2.2)	an electrical current decreasing exponentially with time.

I_{osc}	(3.2.2.2)	an electrical current varying in time as a damped sinusoid.
I_w	(7.2)	when a seismometer drives two galvanometers simultaneously, I_w refers to the current through that galvanometer whose output is being recorded.
i	(7.2)	current through a circuit, or portion of a circuit.
i_1	(3.2.2.2)	current at time t_1.
i_2	(3.2.2.2)	current at time t_2.
i_3	(4.4.1)	a prescribed dc current into a galvanometer used in establishing the performance of a device for converting coil angular position into voltage.
i_ℓ	(4.3.2)	current through an electrical inductor simulating the suspension stiffness of a galvanometer. This current is proportional to the instantaneous angular displacement of the coil.
j	(2.8)	$\sqrt{-1}$
K	(4.2)	moment of inertia of the galvanometer coil. The angular equivalent of M_x.
K_N	(7.2)	when a seismometer drives two galvanometers simultaneously, K_N refers to the moment of inertia of that galvanometer whose output is not being recorded.
K_w	(7.2)	when a seismometer drives two galvanometers simultaneously K_w refers to the moment of inertia of that galvanometer whose output is being recorded.
k	(2.1.1)	the designator of a push rod into a Lagrangian black box.
	(4.4.1)	the proportionality constant, relating output voltage to mirror deflection, of a device for producing a voltage proportional to galvanometer deflection.

k	(8.2)	Boltzmann's constant, approximately 1.37 x 10^{-23} joule/deg.
L	(2.6)	The self inductance of the instrument coil as measured with the mass blocked.
L_1	(2.8.2)	the self inductance of one coil of the instrument as measured with the mass blocked.
L_2	(2.8.2)	the self inductance of a second coil of the instrument as measured with the mass blocked.
L_3	(4.2)	the electrical inductance of the galvanometer coil plus series inductance (if any) in the galvanometer circuit.
L_x	(3.2.4)	numerically G^2/S. The electrical inductance equivalent to the effective stiffness of the suspension in the electrical analog of a seismometer.
L_θ	(4.3.2)	numerically G_3^2/U. The electrical inductance equivalent to the stiffness of the suspension of the galvanometer in the electrical analog.
$\underset{\sim}{\mathcal{L}}$	(2.1.1)	the Lagrangian function. The difference between the kinetic and potential energies associated with a Lagrangian black box. T-V.
ℓ_m	(2.3)	length of the magnet in the seismometer.
M	(3.1)	the mass of the bob. In this and subsequent chapters the assumption is made that $M_x = M_{xy}$. To preserve the distinction would be more precise but tend to obscure the relations deemed to be important.
M_{12}	(2.8.2)	the mutual inductance between two real coils of the instrument.
M_x	(2.4.1)	inertia (apparent mass) associated with $\dot{x}^2$, independent of $\dot{y}$, for computing mechanical kinetic energy. The linear equivalent of K.

M_{xy}	(2.4.1)	inertia (apparent mass) associated with the product $\dot{x}\dot{y}$ for computing mechanical kinetic energy.
M_y	(2.4.1)	inertia (mass) associated with $\dot{y}^2$, independent of $\dot{x}$, for computing mechanical kinetic energy.
m	(3.2.2.1)	the mass used in a weight-lift calibration.
	(9.4.2)	slope of a straight line obtained by rectification of data taken during measurements on a seismometer.
n	(2.2.3)	number of turns in a coil of the instrument.
P	(8.4)	ambient atmospheric pressure around the inertial mass.
Q_1'	(2.1.1)	one of the forces on a push rod associated with a constraint within a Lagrangian black box.
Q_k	(2.1.1)	external force applied to the k^{th} push rod into a Lagrangian black box.
Q_x	(2.2.2)	The generalized force associated with the coordinate of bob motion.
q_1	(2.1.1)	first coordinate of a constraint within a Lagrangian black box. A constraint may be considered as a push rod not accessible from the outside.
q_k	(2.1.1)	generalized coordinate, the position of the k^{th} push rod into a Lagrangian black box.
$\dot{q}_k$	(2.1.1)	velocity of the k^{th} push rod into a Lagrangian black box.
R	(6.3)	the series loop resistance of a direct coupled seismometer-galvanometer combination.
R_o	(9.4.2)	Load resistor across the main coil of a seismometer.
R_1	(2.8.2)	electrical resistance of one real coil of the instrument.
	(5.2)	when a seismometer and galvanometer are connected by a resistive T pad the resistance of the input arm of the T pad plus the resistance of the seismometer coil is indicated by R_1.

R_1	(7.4.1)	resistor of the first of two final RC filters. It is associated with C_1 to give a time constant τ_1 and a 3 db point ω_1.
R_2	(2.8.3)	electrical resistance of the second real coil of the instrument.
	(7.4.1)	resistor of the second of two final RC filters. It is associated with C_2 to give a time constant τ_2 and a 3 db point ω_2.
R_3	(4.2)	the electrical resistance of the galvanometer coil plus series resistance (if any).
	(5.2)	when a seismometer and galvanometer are connected by a resistive T pad the resistance of the output arm of the T pad plus the resistance of the galvanometer coil is taken as R_3.
R_c	(2.2.3)	resistance of the coil of the seismometer.
R_g	(4.3.2)	electrical resistance of the galvanometer coil.
R_{in}	(9.4.2)	resistor in series with a calibrator coil, to monitor the driving current.
R_L	(4.3.3)	the electrical resistance of an inductor used with a galvanometer; measurements on the combination permit determination of $\mathcal{R}_g$.
R_s	(5.2)	the resistance of the shunt (common) arm of a resistive T pad connecting a seismometer to a galvanometer.
R_x	(3.2.4)	numerically G^2/r_x. The electrical resistance which is equivalent to the internal losses in the instrument in an electrical analog of the seismometer.
R_θ	(4.3.2	numerically G_3^2/r_θ . The electrical resistance equivalent to the internal losses of the galvanometer in the electrical analog.

$\mathcal{R}_o$	(6.2)	$\mathcal{R}_o \equiv G_1^2/2\omega_o M$. In a seismometer having negligible internal losses and negligible inductance this is the critical damping resistance (CDRX + coil resistance)
$\mathcal{R}_g$	(4.2)	$\mathcal{R}_g \equiv G_3^2/2\omega_g K$. This is the critical damping resistance (CDRX plus coil resistance), of a galvanometer having no internal losses and negligible inductance.
$\mathcal{R}_N$	(7.2)	when a single seismometer drives two galvanometers it is necessary to distinguish between them. $\mathcal{R}_N$ refers to the critical damping resistance of the galvanometer whose output is not being recorded.
$\mathcal{R}_w$	(7.2)	when a seismometer drives two galvanometers simultaneously, $\mathcal{R}_w$ refers to the critical damping resistance of that galvanometer whose output is being recorded.
$\left.\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}\right\}$	(4.4.2)	resistors in a bridge network for determining instrument parameters.
r_x	(2.2.2)	losses within the instrument tending to reduce the velocity of the inertial mass such as residual eddy currents, air damping, etc. Presumed proportional to velocity. The linear equivalent of r_θ.
r_θ	(4.2)	mechanical damping "constant" of the galvanometer. Usually chiefly air damping but would include losses due to eddy currents in a conducting coil form. The rotational equivalent of r_x.
S	(2.6)	the stiffness of an ideal mechanical spring equivalent to the rate of change with displacement of the sum of the various restoring forces on the inertial mass when the seismometer is subjected to no external forces.

s	(4.4.1)	galvanometer sensitivity in amperes per radian of coil motion.
T	(2.1.1)	the kinetic energy associated with a Lagrangian black box.
T	(8.2)	absolute temperature, degrees Kelvin.
T_{mag}	(2.1.2)	magnetic kinetic energy.
T_{mech}	(2.1.2)	mechanical kinetic energy.
t	(2.1.1)	time.
t_1, t_2	(3.2.2.2)	particular values of time.
U	(4.2)	stiffness of the suspension of the galvanometer coil, the angular equivalent of S.
U_N	(7.2)	when a seismometer drives two galvanometers simultaneously, U_N refers to the restoring torque of that galvanometer whose output is not being recorded.
U_w	(7.2)	when a seismometer drives two galvanometers simultaneously U_w refers to the restoring torque of that galvanometer whose output is being recorded.
V	(2.1.1)	the potential energy associated with a Lagrangian black box.
$V_{electrostatic}$	(2.1.2)	electrostatic potential energy associated with a Lagrangian black box.
V_{grav}	(2.2.1)	gravitational potential energy.
V_{mech}	(2.1.2)	mechanical potential energy associated with a Lagrangian black box.
V_s	(2.4.4)	potential energy stored as elastic distortion of the primary and other springs.

W	(3.2.2)	defined as $(\rho - \beta)/\gamma^2 = \beta/(\beta^2 + 1)$. W is introduced as a matter of convenience to explain the construction of Fig. 3.1.
W_1	(5.2)	total power dissipated by a dc emf introduced in one branch of a resistive T pad.
W_{oo}	(2.4.2)	electrical inertia associated with the current I_o in a hypothetical coil equivalent to the instrument magnet. In (2.7) it is shown to be the self-inductance of the simulated coil.
W_{o1}	(2.4.2)	electrical inertia associated with the product of the current in the simulating coil and the current in the real coil. In (2.7) it is shown to be the mutual inductance between the simulating coil and real coil.
W_{o2}	(2.8.2)	electrical inertia associated with the product of the current in the coil simulating the instrument magnet and the current in a second real coil of the instrument. The mutual inductance between the simulated coil and a second coil on the instrument.
W_{11}	(2.4.2)	electrical inertia associated with the current in the real coil. In (2.7) it is shown to be the self inductance of the real coil of the instrument.
W_{12}	(2.8.2)	electrical inertia associated with the product of the currents in two real coils of the instrument. Identified as the mutual inductance between two real coils of the instrument.
W_{22}	(2.8.2)	electrical inertia associated with the current in a second real coil. Identified as the self inductance of a second real coil.
W_3	(5.2)	power dissipated by currents in one branch of a resistive T pad due to a dc emf introduced into the other branch.

X	(2.2.2)	forces applied directly to the inertial mass parallel to the motional axis, as by an electromagnetic calibrator, or in a vertical instrument, air bouyancy. The linear equivalent of Θ.
x	(2.2.2)	linear displacement of the center of gravity of the principal inertial mass, parallel to the motional axis of the instrument, with respect to the frame of the instrument. For vertical instrument taken as positive upward. The linear equivalent of θ.
$\dot{x}$	(2.2.2)	dx/dt time derivative of x. In a vertical instrument taken as positive for upward velocity of the bob with respect to the frame.
$\ddot{x}$	(2.6)	$\frac{d^2x}{dt^2}$, the acceleration of the bob with respect to the frame. In a vertical instrument it is taken positive for increase of upward velocity.
x_o	(2.6)	the equilibrium position of the inertial mass with respect to the instrument frame when the instrument is subjected to no external forces, i.e. the "rest point" of the inertial mass.
Y	(2.2.1)	force exerted on the seismometer frame by the pier. Taken as positive if it tends to produce a positive displacement y.
	(4.4.2)	vertical deflection of an oscilloscope trace, also a switch position, during the measurement of instrument parameters.
Y'	(4.4.2)	vertical deflection of an oscilloscope trace, also a switch position, during the measurement of instrument parameters.
y	(2.2.1)	earth displacement parallel to the motional axis of the instrument. For vertical instrument y is taken positive upward. The linear equivalent of η.
$\dot{y}$	(2.2.1)	dy/dt, earth velocity parallel to the motional axis of the instrument. For a vertical instrument dy/dt is taken as positive upward.

$\ddot{y}$	(2.6)	$\frac{d^2y}{dt^2}$, acceleration of the frame of the instrument with respect to a "stationary" point.
Z	(9.3.5.3)	by opening the electrical loop of a fourth or sixth order system one may consider it as a two terminal network whose electrical impedance is Z.
Z_g	(4.3.2)	the electrical impedance of a galvanometer considered as a two terminal network.
Z_{sc}	(4.3.2)	the electrical impedance of a galvanometer in series with a capacitor.
Z_θ	(4.3.2)	the electrical impedance equivalent to the mechanical portions of a galvanometer when it is considered as a two terminal electrical network.
α	(3.2.3)	the fraction by which a variable may be attenuated, as a voltage by a potentiometer. Used in this sense only in this section.
α	(5.2)	the mean coupling coefficient, a measure of the attenuation provided by a resistive T pad, $\alpha \equiv R_s/\sqrt{(R_1 + R_s)(R_3 + R_s)}$
β	(3.2.2.)	a root of the auxiliary equation for the transient solution of the differential equation for a seismometer connected to a resistive load.
$\left.\begin{matrix}\beta_1\\ \beta_2\\ \beta_3\end{matrix}\right\}$	(3.2.2)	the three roots of the auxiliary equation for the transient solution of the differential equation for a seismometer connected to a resistive load.
γ	(3.2)	defined as $G/\sqrt{LS} = G/\omega_0\sqrt{LM}$. γ is a normalized generator/motor constant.

η	(4.2)	angular position of the galvanometer case, with respect to a "fixed" direction in space. The angular equivalent of y.
Θ	(4.2)	mechanical torque applied directly to the galvanometer coil. The rotational equivalent of X.
θ	(4.2)	angular position of a galvanometer coil with respect to the case. The rotational equivalent of x.
θ_d	(6.4)	galvanometer deflection for a direct coupled system for comparison with the output of a T pad coupled system subjected to the same excitation.
θ_N	(7.2)	when a seismometer drives two galvanometers simultaneously, θ_N refers to the angular position of that galvanometer coil which is not being recorded.
θ_T	(6.4)	galvanometer deflection for a T pad coupled system, for comparison with the output of a direct coupled system subjected to the same excitation.
θ_w	(7.2)	when a seismometer drives two galvanometers simultaneously, θ_w refers to the angular position of that galvanometer coil which is being recorded.
θ_{ω_o}	(6.3)	angular deflection of the galvanometer mirror at ω_o.
θ_{ω_g}	(6.3)	angular deflection of the galvanometer mirror at ω_g.
λ	(6.3.1)	ratio to critical damping, fraction of critical damping. In the special cases where $\lambda_o = \lambda_g = \lambda_m$ no point is served by preserving the subscript.

λ_o	(6.2)	fraction of critical damping, or ratio to critical damping, of the seismometer due to the circuit resistance as seen by the seismometer (includes coil resistance of the seismometer).
λ_g	(4.2)	fraction of critical damping, or ratio to critical damping, of the galvanometer due to the circuit resistance as seen by the galvanometer (includes coil resistance).
λ_m	(6.2)	mean ratio to critical damping, the geometric mean of the fraction of critical damping of seismometer and galvanometer. $\lambda_m \equiv \sqrt{\lambda_o \lambda_g}$.
λ_N	(7.2)	when a seismometer drives two galvanometers simultaneously it is necessary to distinguish between them. λ_N refers to the ratio to critical damping of the galvanometer whose output is not being recorded.
λ_w	(7.2)	when a seismometer drives two galvanometers simultaneously λ_w refers to the fraction of critical damping which the circuit resistance presents to that galvanometer whose output is being recorded.
μ_o	(2.3)	permeability of the core upon which is wound the coil simulating the instrument's magnet.
μ_r	(2.3)	the "reversible" permeability of the magnet of the instrument, as used.
ν	(3.2.2.1)	when two of the three values of β are complex they may be written as $\sigma + j\nu$ and $\sigma - j\nu$.
ν	(8.3)	The vibration frequency which when multiplied by Planck's constant gives the permissible energy increments of a system.
ρ	(3.2)	defined as $R/\omega_o L$. ρ is the ratio of resistive to inductive impedance at the open circuit frequency of the seismometer.

ρ_a	(8.4)	density of air surrounding the seismic mass.
ρ_m	(8.4)	effective density of the inertial mass.
ρ_x	(3.2)	defined as $r_x/\omega_o M$. ρ_x is the normalized mechanical damping coefficent.
σ	(3.2.2.1)	when two of the three values of β are complex they may be written as $\sigma + j\nu$ and $\sigma - j\nu$.
τ	(3.2)	defined as $\omega_o t$. τ is a normalized time variable. When measured in this unit any seismometer has an open circuit period of 2π.
τ_1	(7.4.1)	time constant of the first of two final RC filters.
τ_2	(7.4.1)	time constant of the second of two final RC filters.
Φ	(2.2.3)	magnetic flux linked with a coil of the instrument.
	(6.3.1)	a normalized angular frequency defined as $\Phi \equiv \omega/\omega_m$.
Φ_o	(2.2.4)	the magnetic flux linked with the simulated coil replacing the permanent magnet in the instrument.
Φ_1	(2.8.2)	the magnetic flux linked with the first real coil of the instrument.
Φ_2	(2.8.2)	the magnetic flux linked with the second real coil of the instrument.
Ψ	(7.4.1)	The phase angle by which the voltage at the recorder leads earth velocity.
Ψ_x	(3.2.1.2)	The phase angle by which the system output leads force applied directly to the seismic mass.
Ψ_y	(3.2.1.1)	The phase angle by which the output current leads earth displacement.

ω	(2.8.3)	2π times frequency in cps, angular frequency, frequency of a sinusoidal variable in radians per second.
ω_o	(2.8.3)	defined as $\sqrt{S/M_x}$, called the natural angular frequency of the instrument. It is the frequency, in radians per second, at which the inertial mass would oscillate after an impulse if it had no internal damping, when disconnected from an electrical circuit.
ω_1	(7.4.1)	3 db point of the first of two final RC filters, frequency at which the capacitative and resistive impedances of the filter elements are equal.
ω_2	(7.4.1)	3 db point of the second of two final RC filters, frequency at which the capacitative and resistive impedances of the filter elements are equal.
ω_g	(4.2)	natural angular frequency of a galvanometer, defined as $\sqrt{U/K}$.
ω_N	(7.2)	when a single seismometer drives two galvanometers it is necessary to distinguish between them. ω_N refers to the natural angular frequency of the galvanometer whose output is not being recorded. $\omega_N \equiv \sqrt{U_N/K_N}$
ω_h	(6.3.1)	angular frequency at which the high frequency asymptote to the amplitude-frequency response curve intersects the midband tangent.
ω_ℓ	(6.3.1)	angular frequency at which the low frequency asymptote to the amplitude-frequency response curve intersects the midband tangent.
ω_m	(6.2)	the geometric mean of the two instrument natural angular frequencies. $\omega_m \approx \sqrt{\omega_o \omega_g}$.

ω_{pc}	(4.3.2)	that angular frequency at which the parallel combination of a capacitor and galvanometer shows no reactive component of impedance.
ω_{PL}	(4.3.2)	that angular frequency at which the parallel combination of an inductor and galvanometer shows no reactive component of impedance.
ω_s	(3.2.1.1)	defined as $2\pi f_s$. The "short-circuit" angular frequency of the seismometer.
ω_{sc}	(4.3.2)	that angular frequency at which a capacitor and galvanometer in series show no reactive component of impedance.
ω_v	(6.2)	that angular frequency of earth motion at which the galvanometer displacement is in phase with earth velocity.
ω_w	(7.2)	when a seismometer drives two galvanometers simultaneously, ω_w refers to the natural angular frequency of that galvanometer whose output is being recorded. $\omega_w \equiv \sqrt{U_w/K_w}$

10.3 References

Chakrabarty, S. K.

Response Characteristics of Electromagnetic Seismographs and Their Dependence on the Instrumental Constants.

Bul. Seism. Soc. Amer. 39, 205-218, (1949).

C.	J & M	Remarks
D	r_x	identical
D_1	r_θ	"
g	G_3	Possible ambiguity of sign
K_1	K	identical
M	M	identical
R	R_1	"
r	R_3	"
S	R_s	"
t	t	"
u_1	U	"
x	x	"
y	y	"
θ	θ	"
φ_m	Φ_o	"
ω	ω_o	"
ω_e	ω	generally identical
ω_g	ω_g	identical

C.	J & M	Remarks
$K\Phi_m/\ell$	G_1	approximately equal
$S^1 - \frac{\Phi_m^2(\ell^2 + 3r_o^2)}{a\ell(\ell^2 - r_o^2)}$	S	J & M found no need to separate the effective stiffness into its components
$S^2/(R + S)(r + S)$	α^2	Why C. should state that "For efficiency and sensitivity" α^2 "is always less than about 0.05" is obscure, cf. p. 211 line 6.

Como, Neil M.

Private Communication dated June 1955

C.	J & M	Remarks
a_o	$\omega_o^3\rho$	identical
a_1	$\omega_o^2\rho$	"
a_2	$\omega_o\rho$	"
ω_s	$\omega_o f_s$	Como's ω_s approaches J & M's $\omega_o f_s$ as the circuit resistance approaches zero.

Como, Neil M.

Theory and Analysis of the Benioff Variable Reluctance Seismometer-Galvanometer System.

Master's Thesis, Rensselaer Polytechnic Institute, Troy, New York June 1955. Library No. C-73.

C.	J. & M	Remarks
a_o	$\omega_o^3 \rho$	identical
a_1	$\omega_o^2 f_s^2$	identical
a_2	$\omega_o \rho$	"
D	r_x	"
D_1	r_θ	"
G	G_1	"
g	G_3	"
I	I	"
K	S	"
L_t	L	"
M	M	"
M_1	K	"
n	n	"
R_1	R_c	"
R_c	R_1	Probably identical
R_g	R_3	" "
R_s	R_s	identical

C	J & M	Remarks
r	R_g	identical
U_1	U	"
x	x	"
$\dot{x}$	$\dot{x}$	"
$\ddot{x}$	$\ddot{x}$	"
y	y	"
$\dot{y}$	$\dot{y}$	"
$\ddot{y}$	$\ddot{y}$	"
θ	θ	"
λ_g	λ_g	"
ϕ_m	ϕ_m	similar
ω	ω	identical
ω_g	ω_g	identical, except in Como's Figure XXVII-b.
ω_n	ω_o	identical
ω_s	$\omega_o f_s$	Como's ω_s approaches J & M's $\omega_o f_s$ as the circuit resistance approaches zero.

Cosson, J.

Équivalent Électrique d'un Galvanomètre à Cadre Mobile

Mesures, 143, 127-130,(April 1949)

The following are identical.

C.	J & M
A	r_θ
B	G_3/U
C	U
i	I_3
K	K
L	L_θ
t	t
T	$2\pi/\omega_g$
U	E_3
y	i_ℓ
γ	C_θ
θ	θ
ρ	R_g
ϕ	G_3
r	r_θ

Coulomb, J.

Accroissment de la période d'un galvanomètre par emploi d'un condensateur.

Mesures, 179, 221-224 (1952)

C.	J & M	Remarks
C	C_e	identical
f	r_θ	"
g	G_3	"
i	I_3	identical, except that J & M may use i where no ambiguity can exist.
k	K	identical
r	R	"
u	U	"
θ	θ	"
ω	ω_g	"

Coulomb, J. and Grenet, G.

Nouveaux Principes de Construction des Séismographes Électromagnétique.

Annales de Physique, 11, 321-369 (1935)

C & G	J & M	Remarks
D	r_x	analogous
d	r_ϑ	identical
G	G_1	analogous
g	G_3	identical
K	M	analogous
k	K	identical
R	R_1	identical when a T pad is used.
	R	identical when instruments are direct coupled
S	R_3	identical
U	S	analogous
u	U	identical
α	λ_g	identical when internal losses of galvanometer may be neglected.
β	λ_o	identical when internal losses of seismometer may be neglected.
θ	θ	identical
Σ	$2\lambda_m\omega_m$	identical when internal losses of instruments may be neglected.
Φ	x	analogous
Ω_o	ω_o	identical
ω'	ω_m	identical
ω_o	ω_g	identical

Dennison, A. T.

The Design of Electromagnetic Geophones

Geophysical Prospecting, 1, 1-28 (1953)

J & M believe that Fig. 2 (b) is in error, that the intended input is as shown in Fig. 6 (b), c.f. D's text p. 7.

D	J & M	Remarks
C	C_x	identical
h	λ_o	identical when internal losses are negligible
K	G_1	identical
L	L	identical
L'	L	identical
M	M_x	identical
R	R	identical, i.e. J & M's total circuit resistance in section 3
R'	R_c	identical
r	r_x	identical
r'	R_x	identical
S	S	identical
w_o	ω_o	identical
w_1	$\omega_s\omega_o$	identical
x	x or y	D. uses x with both meanings

Eaton, J. P.

Theory of the Electromagnetic Seismograph

Bull. Seism. Soc. Amer. 47, 37-75 (1957)

E.	J & M	Remarks
D	r_x	analogous
d	r_θ	identical
G	G_i	analogous
g	G_3	identical
K	M	analogous
k	K	identical
R	R_1	"
r	R_3	"
S	R_s	"
t	t	"
U	S	analogous
u	U	identical
x	y	identical
α_1	λ_g	identical when E's α_o is negligible
β_1	λ_o	identical when E's β_o is negligible
ν	θ	identical
φ	x	analogous
Ω_o	ω_o	identical
ω'	ω_m	identical
ω_o	ω_g	identical
$S^2/(S^2 + Q^2)$	α^2	identical

Fix, James E.

Theoretical and Observed Noise in a High-Sensitivity Long-Period Seismograph

Bull. Seism. Soc. Amer. 63, 1979-1998 (1973)

Of the sixty three-symbols used in this reference, the following thirty five are identical to those used in this report:

C_x
C_θ
f
G_1
G_3
i
i_3
j
K
L_x
L_θ
M
R
R_g
R_x
R_θ
$\mathcal{R}_o$
$\mathcal{R}_g$
r_x
r_o
S
T
t
U
α
λ_o
λ_g
λ_x
λ_θ
η
θ
ω
ω_g
ω_m
ω_o

The following six identities should also be noted:

F	J&M	Remarks
k	k	identical
R_o	R_c	"
Λ_g	$\lambda_g+\lambda_\theta$	"
Λ_o	$\lambda_o+\lambda_x$	"
τ_g	$2\pi/\omega_g$	"
τ_o	$2\pi/\omega_o$	"

Maxwell, J. C.

A Treatise on Electricity and Magnetism

Article 831; Part IV, Chapter V
3rd Ed. Clarendon Press, Oxford, 1904.

Pomeroy, Paul W. and Sutton, George H.

The Use of Galvanometers as Band-Rejection Filters in Electromagnetic Seismographs.

Bull. of the Seism. Soc. Amer. 50, 1, 135-151, (Jan. 1960

P & S	J & M	Remarks
C	$-M_{xy}/M_x$	identical
C'	$-M_{xy}$	identical
G_o	G_1	identical except for possible ambiguity of sign
G_{fl}	G_N	identical except for possible ambiguity of sign
G_g	G_w	identical except for possible ambiguity of sign
I_o	M_x	used identically
I_{fl}	K_N	identical
I_g	K_w	identical
i	i	identical except for possible ambiguity of sign
j	j	identical
K_o	S	used identically
K_g	U_w	identical

P & S	J & M	Remarks
K_j	U_N	identical when P & S are restricted to a single galvanometer as a filter
R	R_1	identical when T pad is used
	R	identical for direct coupled system
r	R_3	identical
S	R_s	identical
x	x	identical
y	y	identical
ϵ_j	$\lambda_{(\ )}\omega_{(\ )}$	identical
θ	θ_w	identical
ϕ_1	θ_N	identical
ω_j	$\omega_{(\ )}$	identical
$\sigma_o\ \sigma_g$	$4\lambda_o\lambda_g\omega_o\omega_g$	identical

Rybner, Jörgen

Investigations on the Theory of the Galitzin Seismograph

Gerland's Beiträge zur Geophysik, 31, 259-281, (1931).

R.	J & M	Remarks
D	r_x	analogous
d	r_θ	identical
E	E	J & M use various subscripts for special situations.
G	G_1	analogous
g	G_3	identical
I	I_1	identical
i	I_3	identical
K	M	analogous
k	K	identical
n	ω_o	identical
n_1	ω_g	identical
p^n	d^n/dt^n	identical
R	R_1	identical when a T pad is used
	R	when direct coupled J & M lump the resistors together as R.
r	R_g	sometimes identical
	R_3	identical when a T pad is used
	R	when direct coupled J & M lump the resistors together as R.
S	R_s	identical
t	t	identical
U	S	analogous
u	U	identical
x	y	identical

R.	J & M	Remarks
ϵ	$\lambda_o \omega_o$	identical when internal losses of the seismometer may be neglected
ϵ	$\lambda_g \omega_g$	identical when internal losses of the galvanometer may be neglected.
θ	θ	identical
ψ	x	analogous
τ	τ	identical
Q^2/S^2	$(1 - \alpha^2)/\alpha^2$	identical

Rybner, Jörgen

The Determination of the Instrumental Constants of the Galitzin Seismograph in the Presence of Reaction.

Gerland's Beiträge zur Geophysik, 51, 375-401, (1937)

(correction of a numerical error, ibid., 55, 303-313, (1939)

In addition to the notation of the previous paper the following symbols are introduced.

R.	J & M	Remarks
T	$2\pi/\omega_o$	identical
X	X	analogous
X_1	θ	identical
μ	$1 - \lambda_o^2$	identical when internal losses in the seismometer are neglected
σ	α	identical when internal losses of both the seismometer and galvanometer are neglected

Scherbatskoy, S. A. and Neufeld, J.

Fundamental Relations in Seismometry

Geophysics 2, 188-212 (1937)

S & N	J & M	Remarks
C_{mg}	$1/U$	identical
e	E_1	identical
I_{mg}	K	identical
i	I_1	identical
i_o	I_o	identical
i_g	I_3	identical
K_g	G_3	identical
K_{gm}	G_1	identical
L_o	W_{oo}	identical
M	W_{o1}	identical
m_m	M_x	identical, J & M believe the definition on p. 201 is a misprint.
R	R_c	identical
R_g	R_g	identical
r_m	r_x	identical
r_{mg}	r_θ	identical
s	x	identical
T	T	identical
T_e	T_{mag}	identical
T_m	T_{mech}	identical
V	V	identical
V_m	V_s	identical
θ_g	θ	identical

Silverman, Daniel

The Frequency Response of Electromagnetically Damped Dynamic and Reluctance Type Seismometers

Geophysics 4, 53-68 (1939)

S	J & M	Remarks
h	λ_o	analogous. S. includes internal damping which J & M generally presume zero.
M	M_x	identical
S	S	"
x	x	"
y	y	"
$\dot{y}$	$\dot{y}$	"
ω_o	ω_o	"

Sparks, Neil R. and Hawley, Paul F.

Maximum Electromagnetic Damping of a Reluctance Seismometer

Geophysics 4, 1-7, (Jan. 1939)

S & H	J & M	Remarks
D	$\rho\omega_o$	identical
m	β	"
ω_o	ω_o	"
ω_1	ω_s	"
$\alpha + j\beta$	$\sigma + j\nu$	"

Sutton, George and Oliver, Jack

Seismographs of High Magnification at Long Periods

Ann. de Géophysique, 15, 423-433, (1959)

S & O	J & M	Remarks
K	M_{xy}/M_x	analogous
S	R_s	identical
Y	y	identical
θ	θ	identical
$\mathfrak{X}$	x	identical
ω	ω_o	identical
ω_e	ω	identical except that J & M do not restrict the running variable to earth motion
ω_g	ω_g	identical
$R_o + R$	R_1	identical
$r + r_g$	R_3	identical

Wenner, Frank

A New Seismometer Equipped for Electromagnetic Damping and Electromagnetic and Optical Magnification.

Natl. Bur. Stand. J. Research 2, 963-999, (1929) RP66

W.	J & M	Remarks
D	r_x	analogous
d	r_θ	identical
G	G_1	analogous
g	G_3	identical
$\underline{g}$	g	identical
H	H	identical
j	j	identical
K	M	analogous
k	K	identical
M	M	identical
Q^2	$R_1R_s + R_1 R_3 + R_3R_s$	identical
S	R_s	identical
T_o	$2\pi/\omega_o$	identical
t	t	identical
U	S	analogous
u	U	identical
X	y	identical
θ	θ	identical when W. uses θ to refer to the galvanometer armature
ϕ	x	analogous
ω	ω	identical although W. restricts his use to earth motion.

Wenner, F. and McComb, H. E.

The Galitzin Seismometer. Discrepancies Between the Galitzin Theory and The Performance of a Wilip-Galitzin Seismometer.

Bull. Seism. Soc. Amer. 26, 317-322 (1936).

W & M	J & M	Remarks
D	r_x	analogous
d	r_θ	identical
G	G_1	analogous
g	G_3	identical
j	j	identical
K	M_x	analogous
k	K	identical
r	R	identical
t	t	identical
U	S	analogous
u	U	identical
x	y	identical
θ	θ	identical
ϕ	x	analogous
ω	ω	identical
ω_o	ω_o & ω_g	W & M are concerned only with systems for which $\omega_o = \omega_g$.
ML	M_{xy}	analogous

Whittaker, E. T.

A Treatise on the Analytical Dynamics of Particles. Section 27, Chapters II and VIII. 4th Ed. Dover Publications, 1944.

Willmore, P. L.

The Theory and Design of Two Types of Portable Seismograph

Mon. Not. Royal Astro. Soc., Geophys. Suppl. 6, 2, 129-138, (1950)

W	J & M	Remarks
A	y	identical
I	K	W. uses I as the moment of inertia of a galvanometer mirror. It therefore represents a lower limit for J & M's K when optical sensing is used.
M_s	M	identical
θ	θ	identical
ω	ω	identical
ω_o	ω_g	identical
ω_s	ω_o	identical

Willmore, P. L.

The Application of the Maxwell Impedance Bridge to the Calibration of Electromagnetic Seismographs.

Bull. Seism. Soc. Amer. 49, 1, 99-114, (Jan. 1959)

The following symbols appear to be identical.

W.	J & M
D	r_x
j	j
K	$-G_1$
M	M
S	S
X	y
x_M	x
θ	θ
ω	ω
ω_M	ω_o

Wolf, Alfred

The Limiting Sensitivity of Seismic Detectors

Geophysics, 7, 2, 115-122, (Apr. 1942)

W.	J & M	Remarks
A	y	identical for sinusoidal earth motion
k	k	identical
M	M	identical
T	T	identical where J & M use T to refer to absolute temperature
x	x	identical
ω	ω	identical
ω_o	ω_o	identical
ℓH	G_1	identical

Worrell, Francis T.

On the Calibration of the Wenner Seismograph.

Bul. Seism. Soc. Amer., 32, 31-48, (1942).

W	J & M	Remarks
D	r_x	analogous
d	r_θ	identical
G	G_1	analogous
g	G_3	identical
K	M	analogous
k	K	identical
$\bar{R}$	$\mathcal{R}_o$	identical when r_x is negligible
R_c	$\mathcal{R}_o - R_c$	identical when r_x is negligible
r	R_g	identical
$\bar{r}$	$\mathcal{R}_g$	identical when r_θ is negligible
r_c	$\mathcal{R}_g - R_g$	identical when r_θ is negligible
S_i	S/G_1	identical
s_i	s	identical
T	$2\pi/\omega_o$	identical
t	$2\pi/\omega_g$	identical
U_1	S	analogous
Ω	ω_o	identical
ω	ω_g	identical

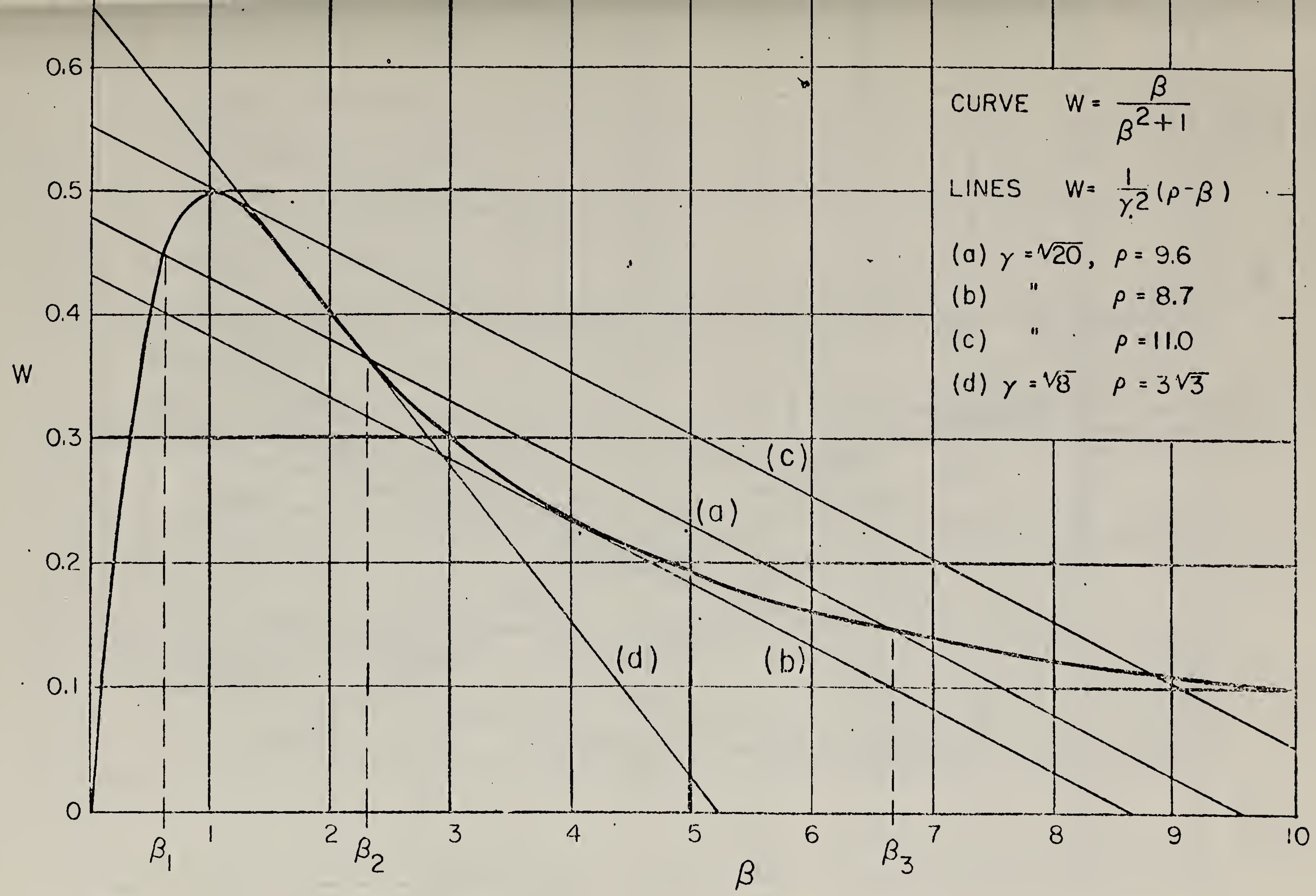


Fig. 3.1a Graph for Obtaining Real Roots of an Algebraic Cubic

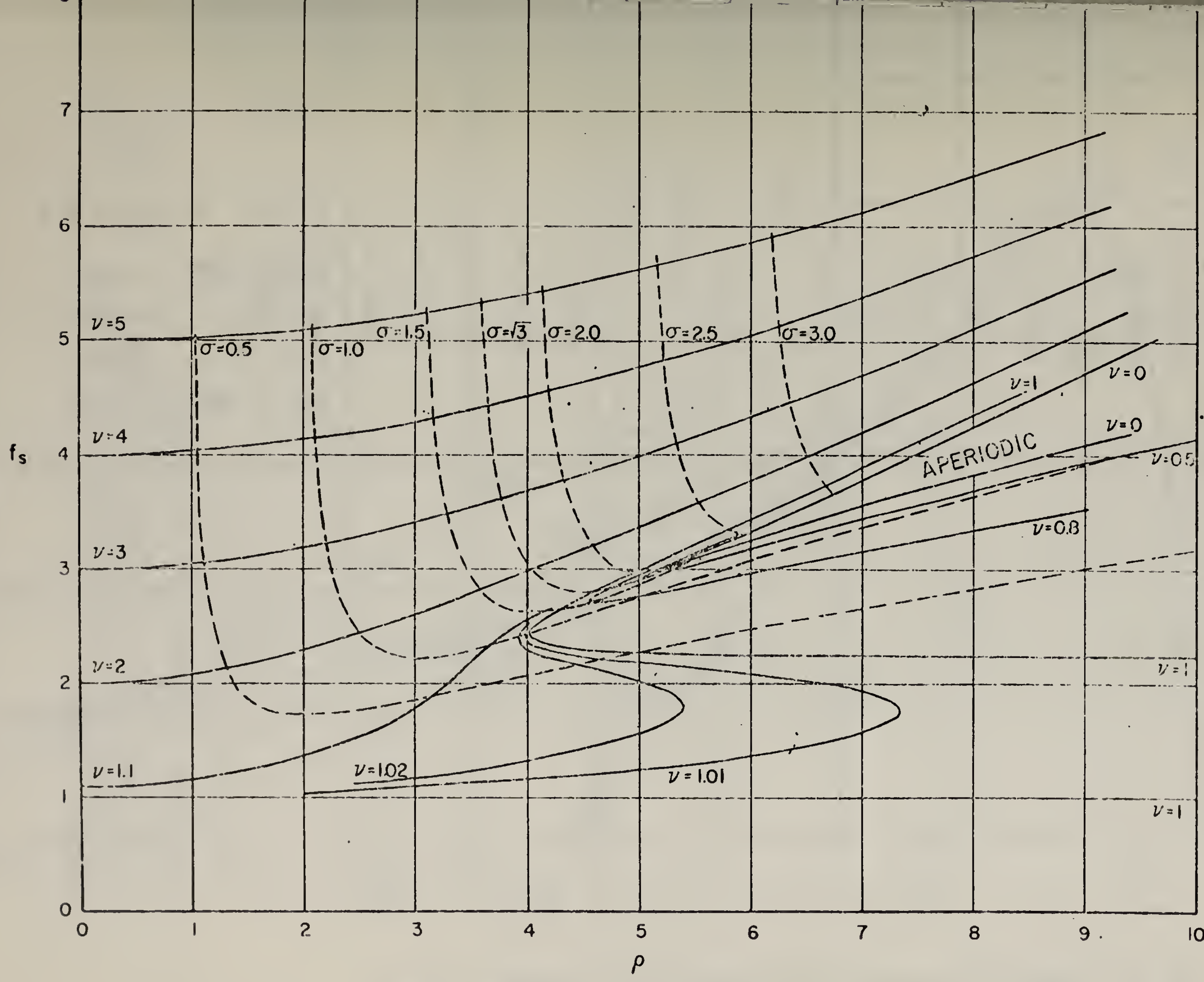


Fig. 31b Plot of σ and ν against f_s and ρ

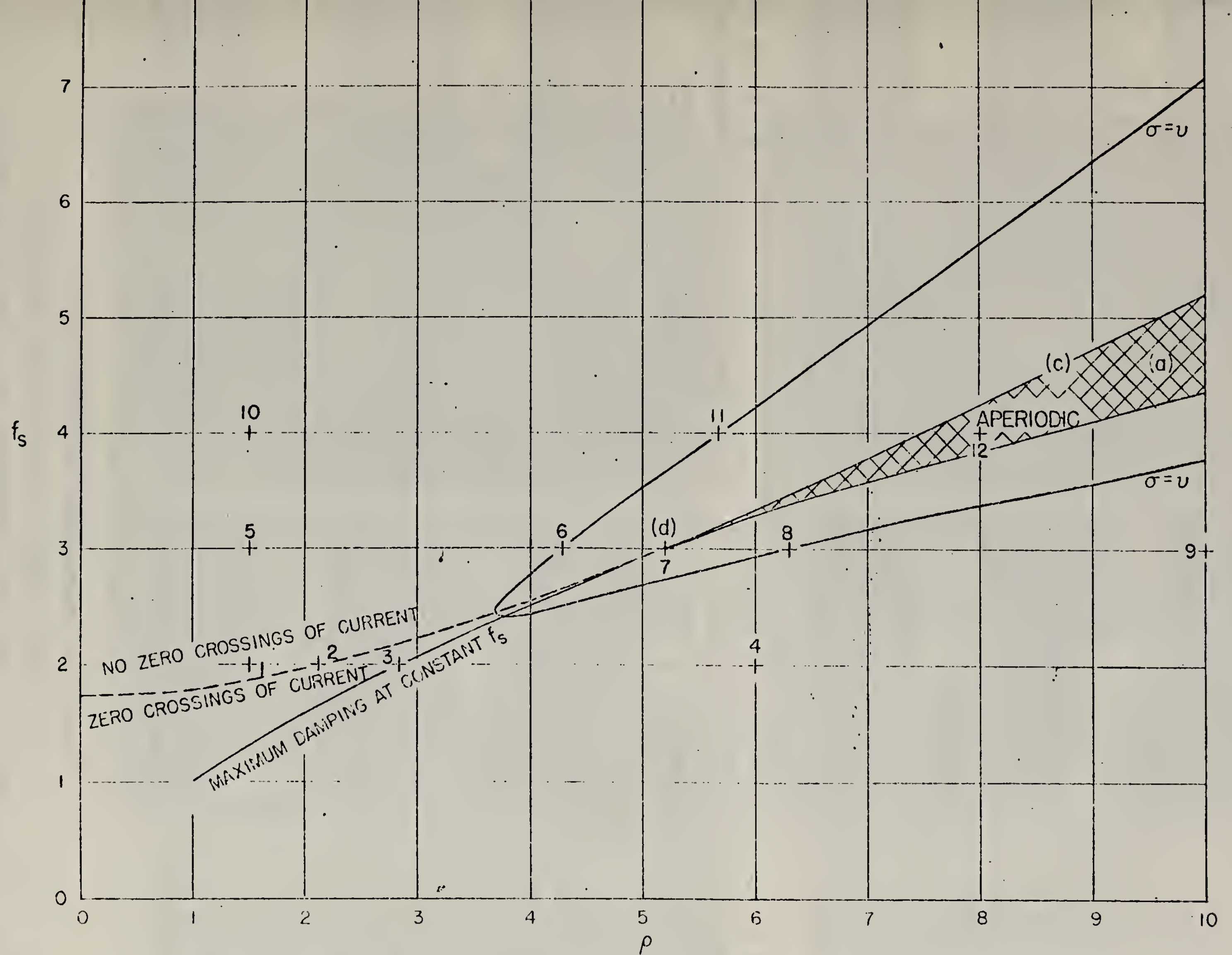
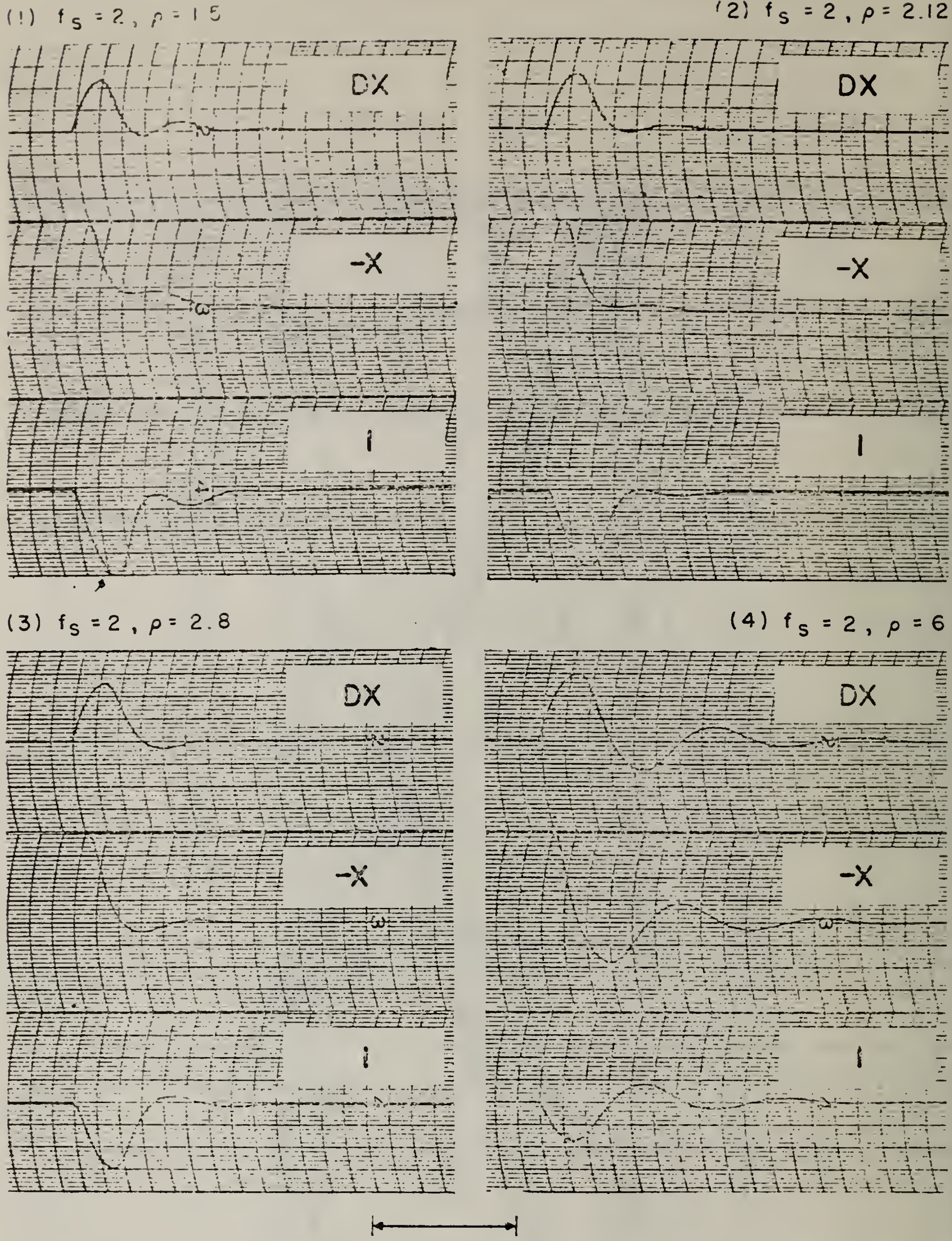


Fig. 3.2 Character of the Transient Response


(1) $f_s = 2,\ \rho = 1.5$

(2) $f_s = 2,\ \rho = 2.12$

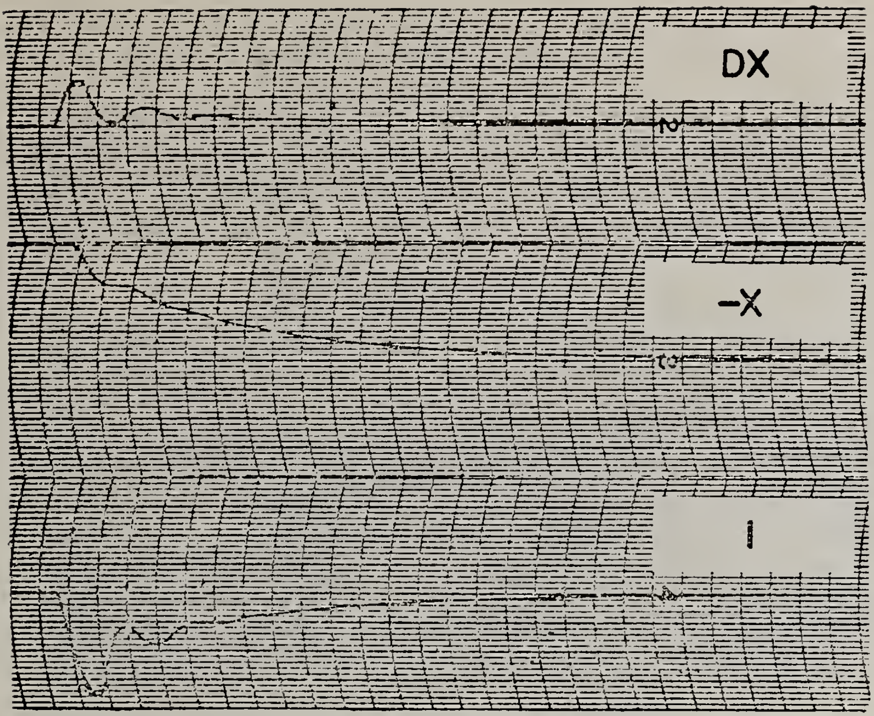
(3) $f_s = 2,\ \rho = 2.8$

(4) $f_s = 2,\ \rho = 6$

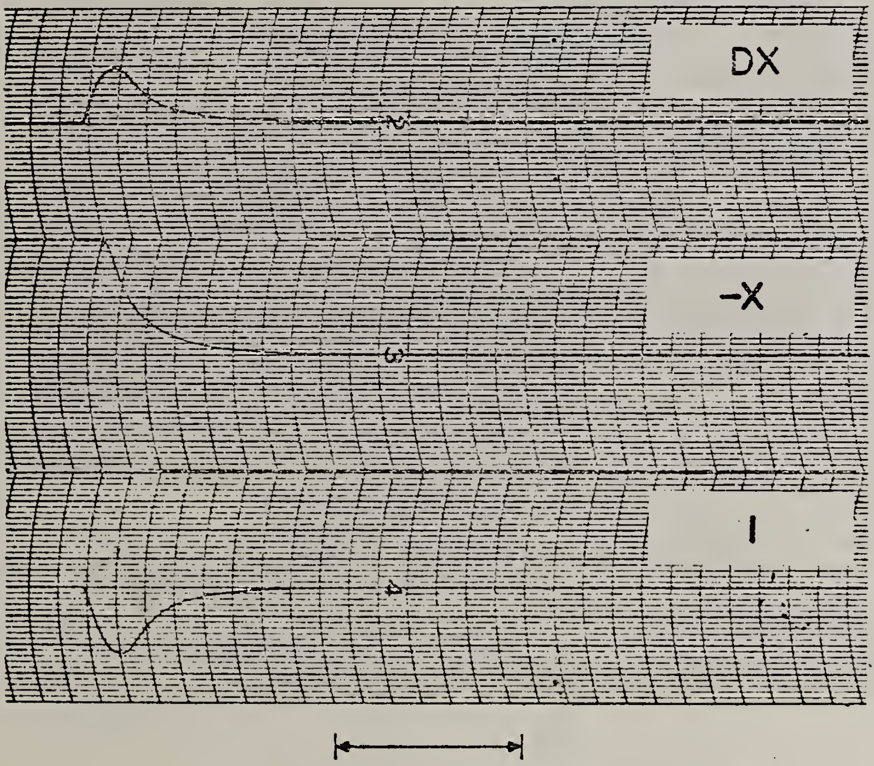
Period of Seismometer on open circuit

Fig. 3.3 Analog Computer Response to Simulated Weight Lift

(5) $f_s = 3$, $\rho = 1.5$



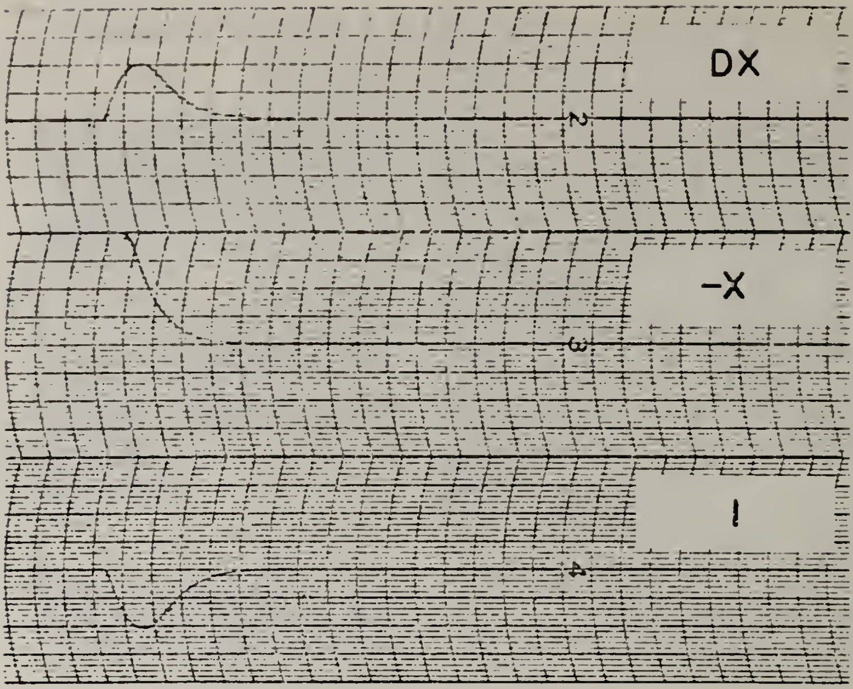
(6) $f_s = 3$, $\rho = 4.28$



Period of Seismometer on open circuit

Fig. 3.4 Analog Computer Response to Simulated Weight Lift

(7) $f_s = 3$, $\rho = 5.2$



(8) $f_s = 3$, $\rho = 6.3$

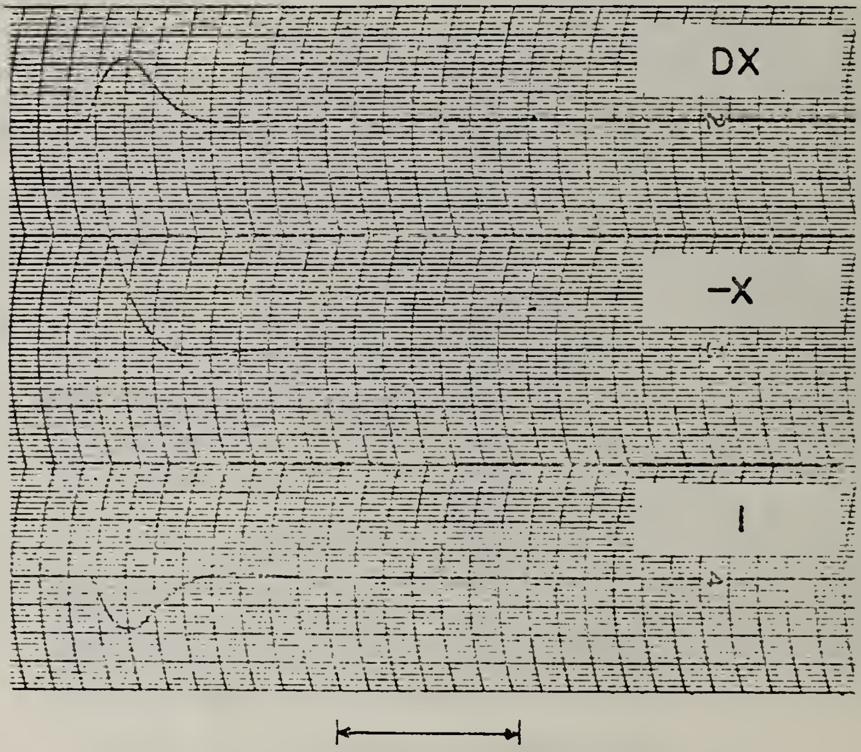


Fig. 3.5 Analog Computer Response to Simulated Weight Lift

(9) $f_s = 3$, $\rho = 10$

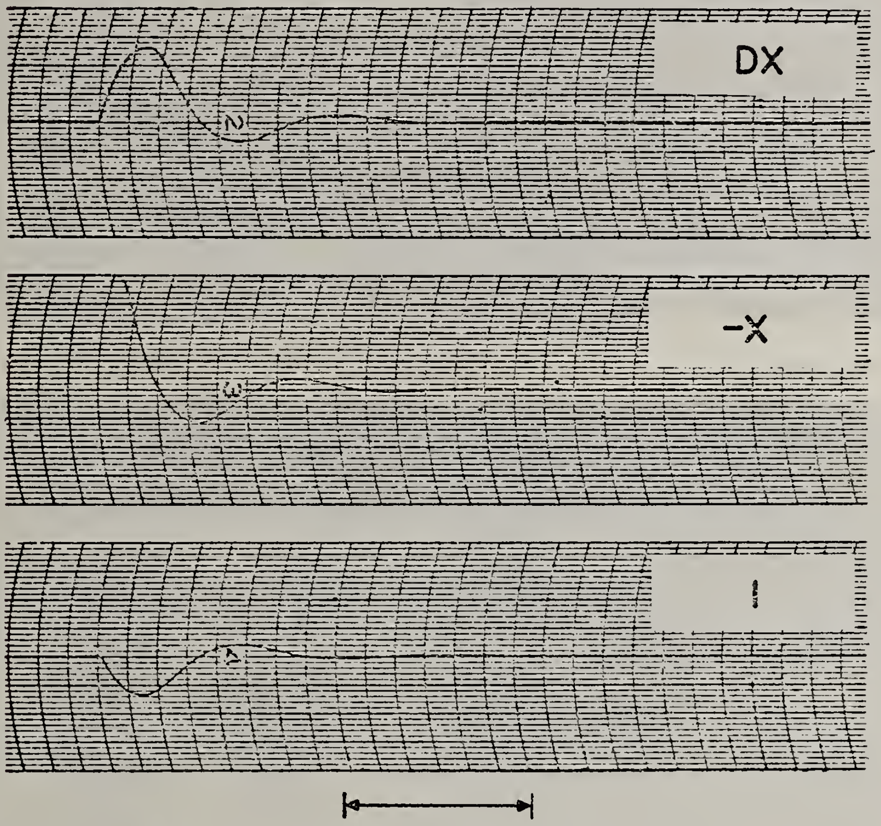
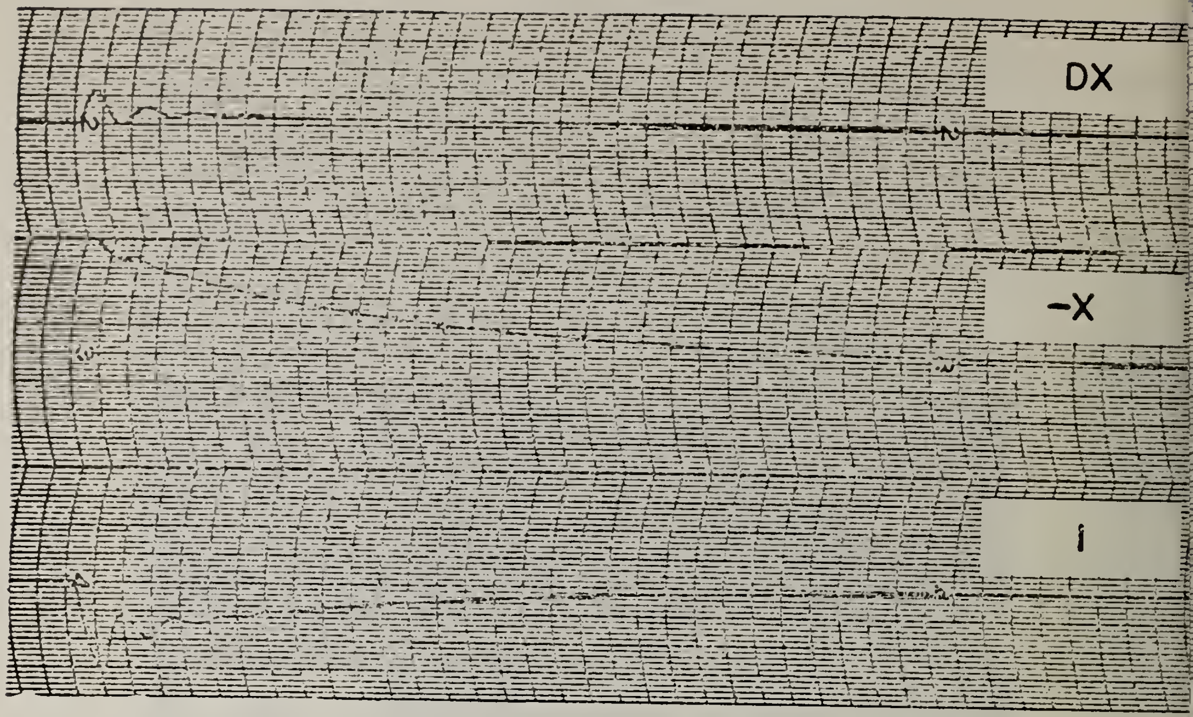
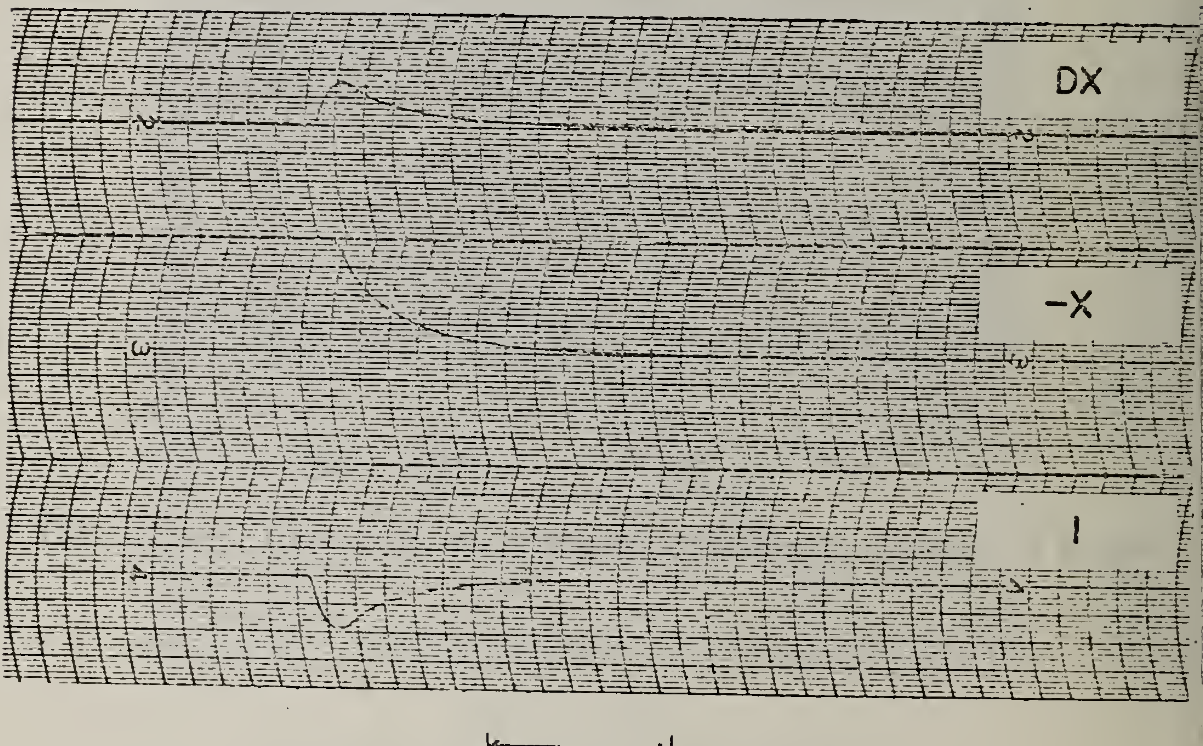


Fig. 3.6 Analog Computer Response to Simulated Weight Lift

(10) $f_s = 4$, $\rho = 1.5$



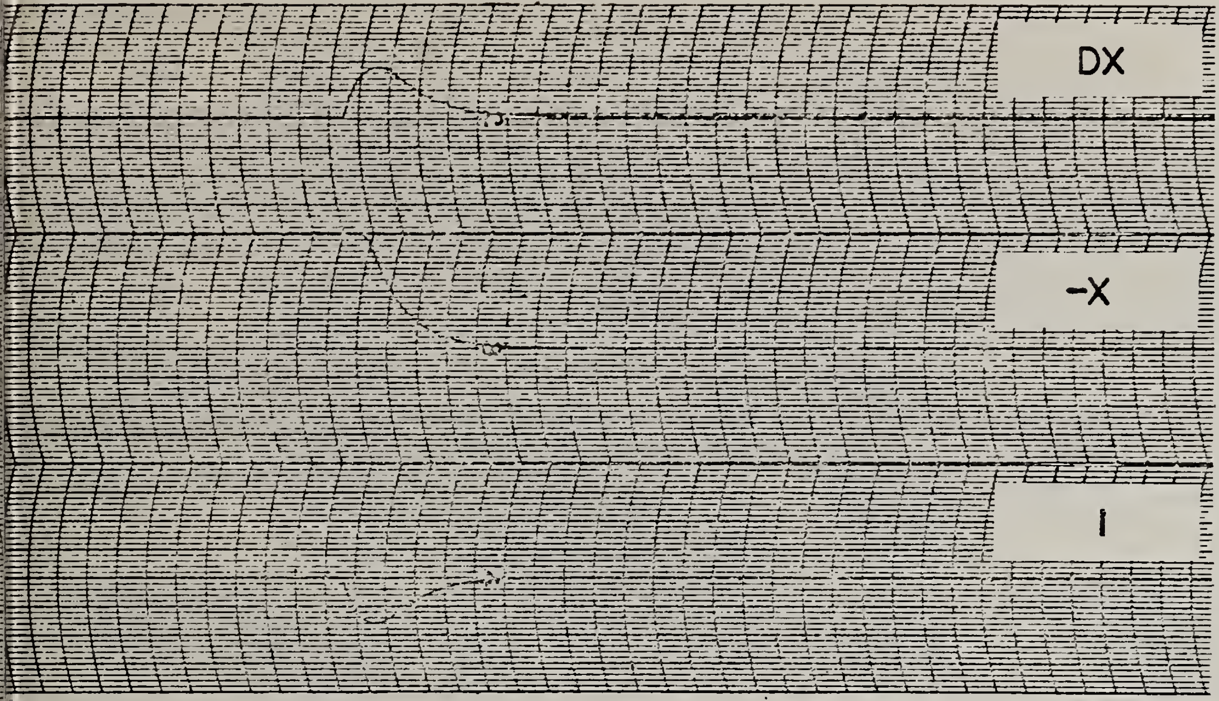
(11) $f_s = 4$, $\rho = 5.67$



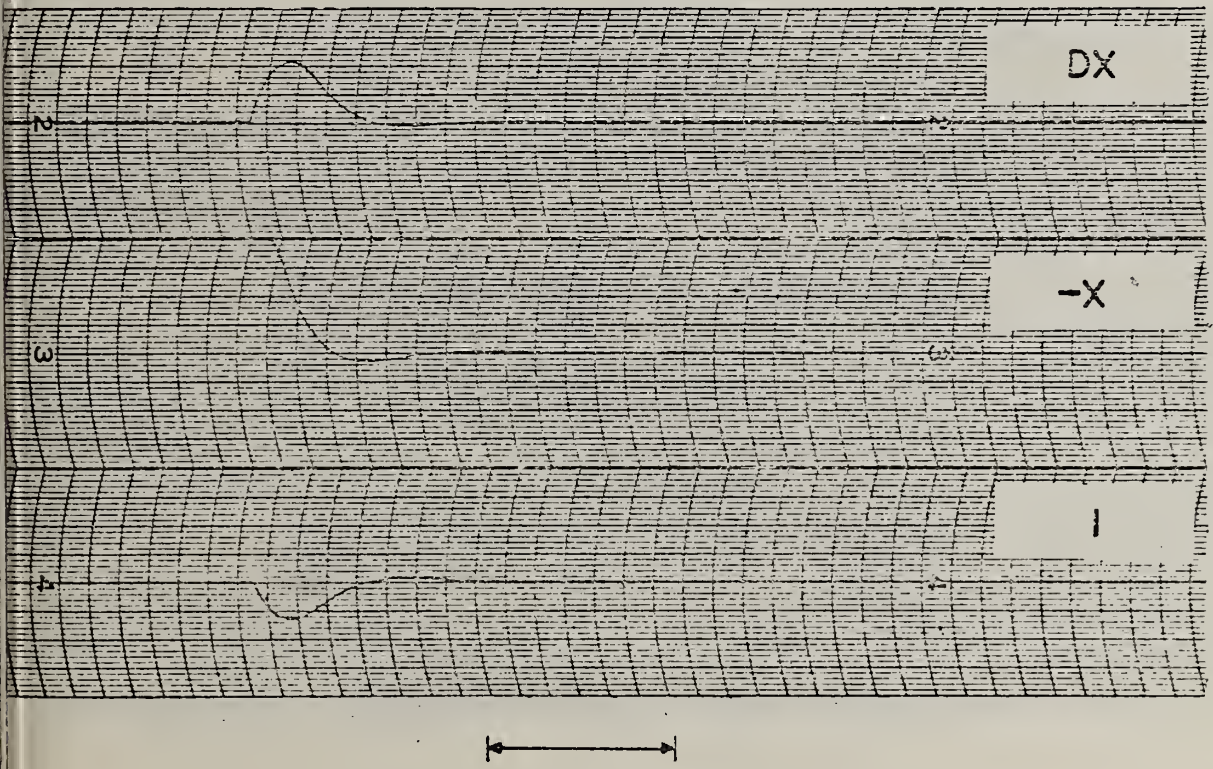
Period of Seismometer on open circuit

Fig. 3.7 Analog Computer Response to Simulated Weight Lift

(12) $f_S = 4$, $\rho = 8.0$



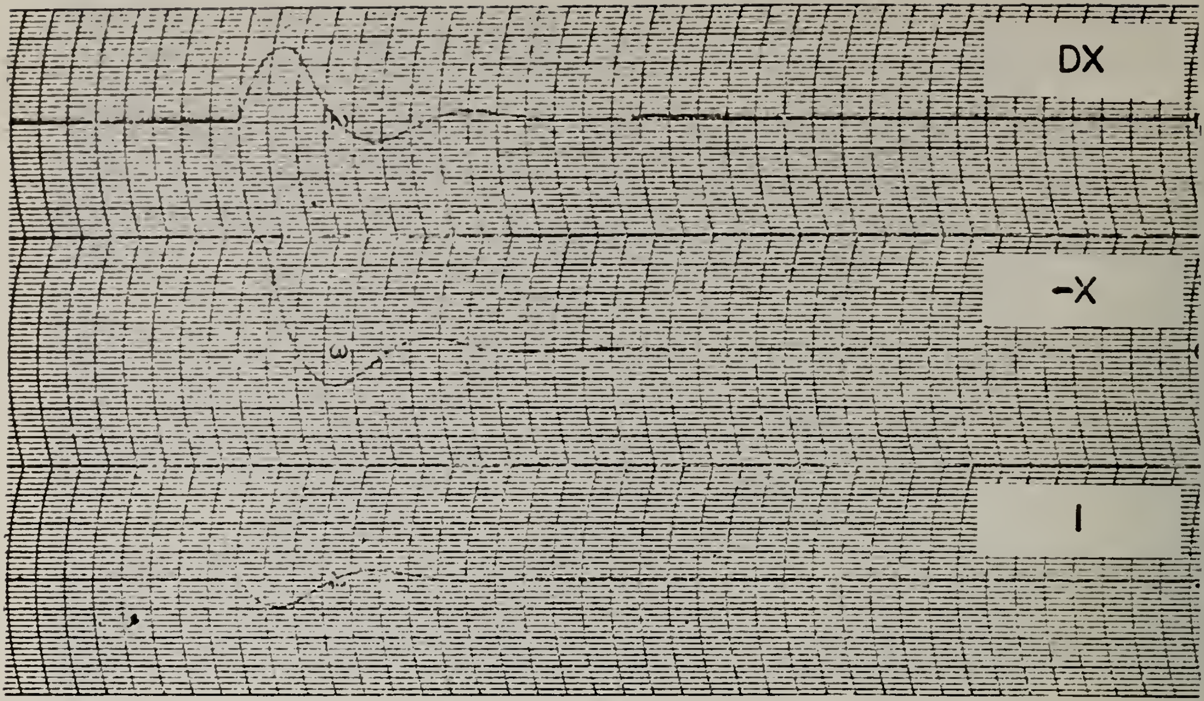
13) $f_S = 4$, $\rho = 11.48$



Period of Seismometer on open circuit

Fig. 3.8 Analog Computer Response to Simulated Weight Lift

(14) $f_s = 4$, $\rho = 20$



(15) $f_S = 4$, $\rho = 50$

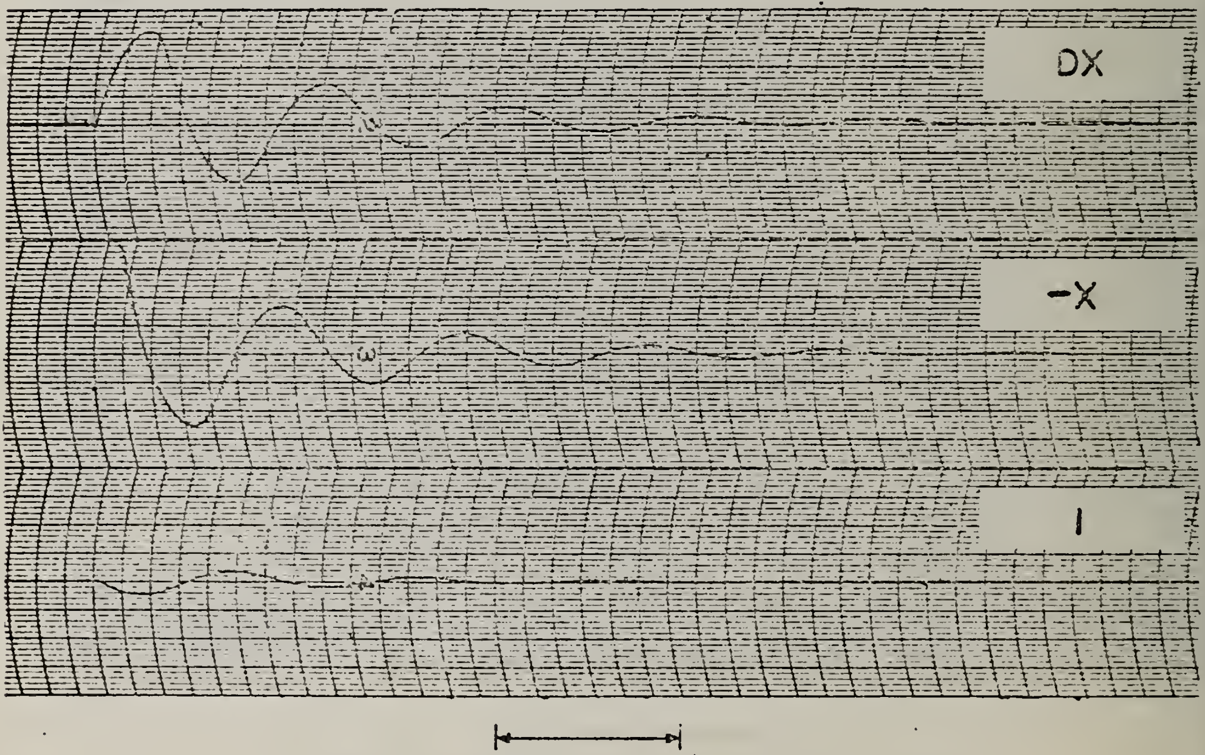


Fig. 3.9 Analog Computer Response to Simulated Weight Lift

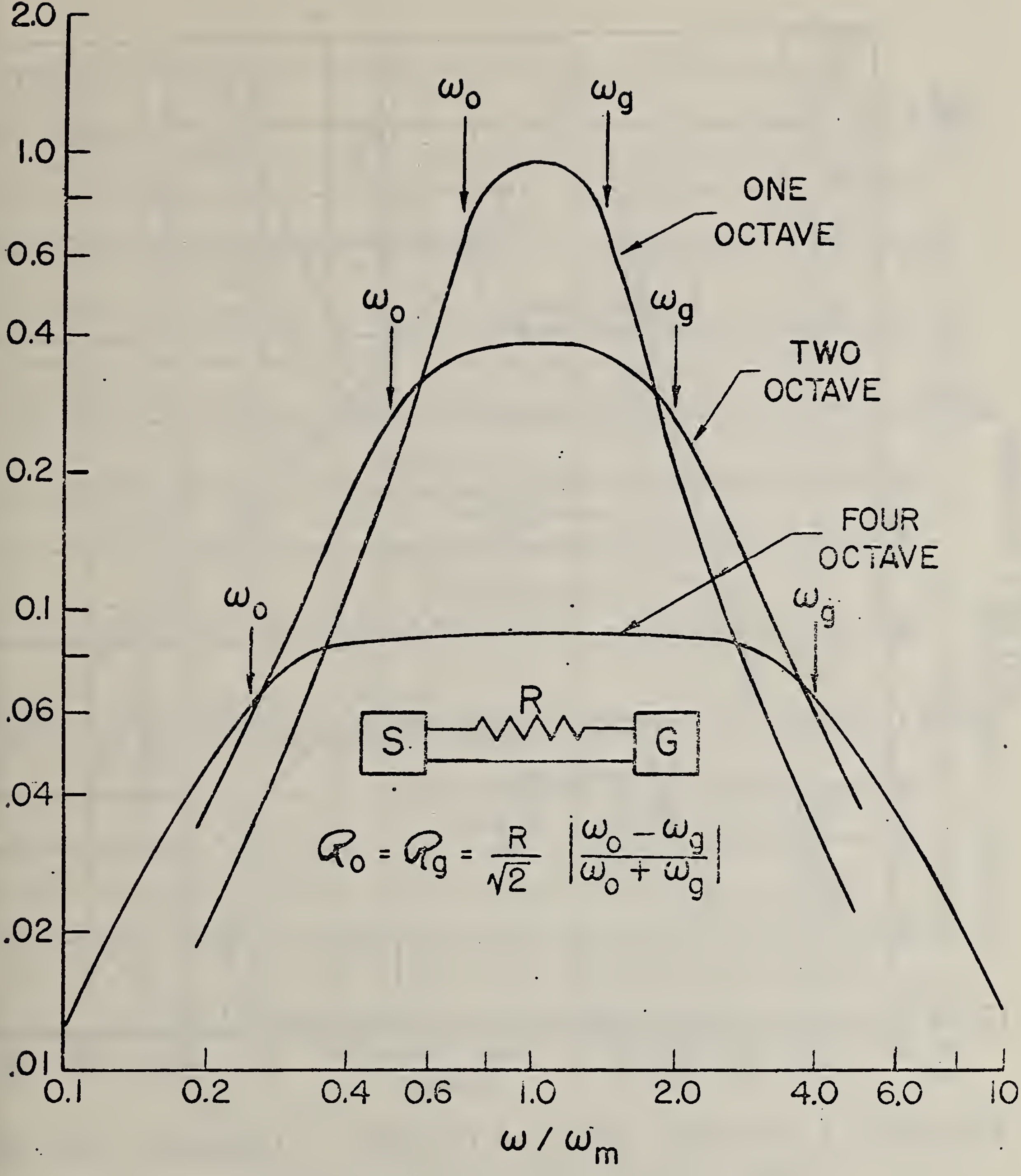


FIG. 6.1 FOURTH ORDER FLAT VELOCITY

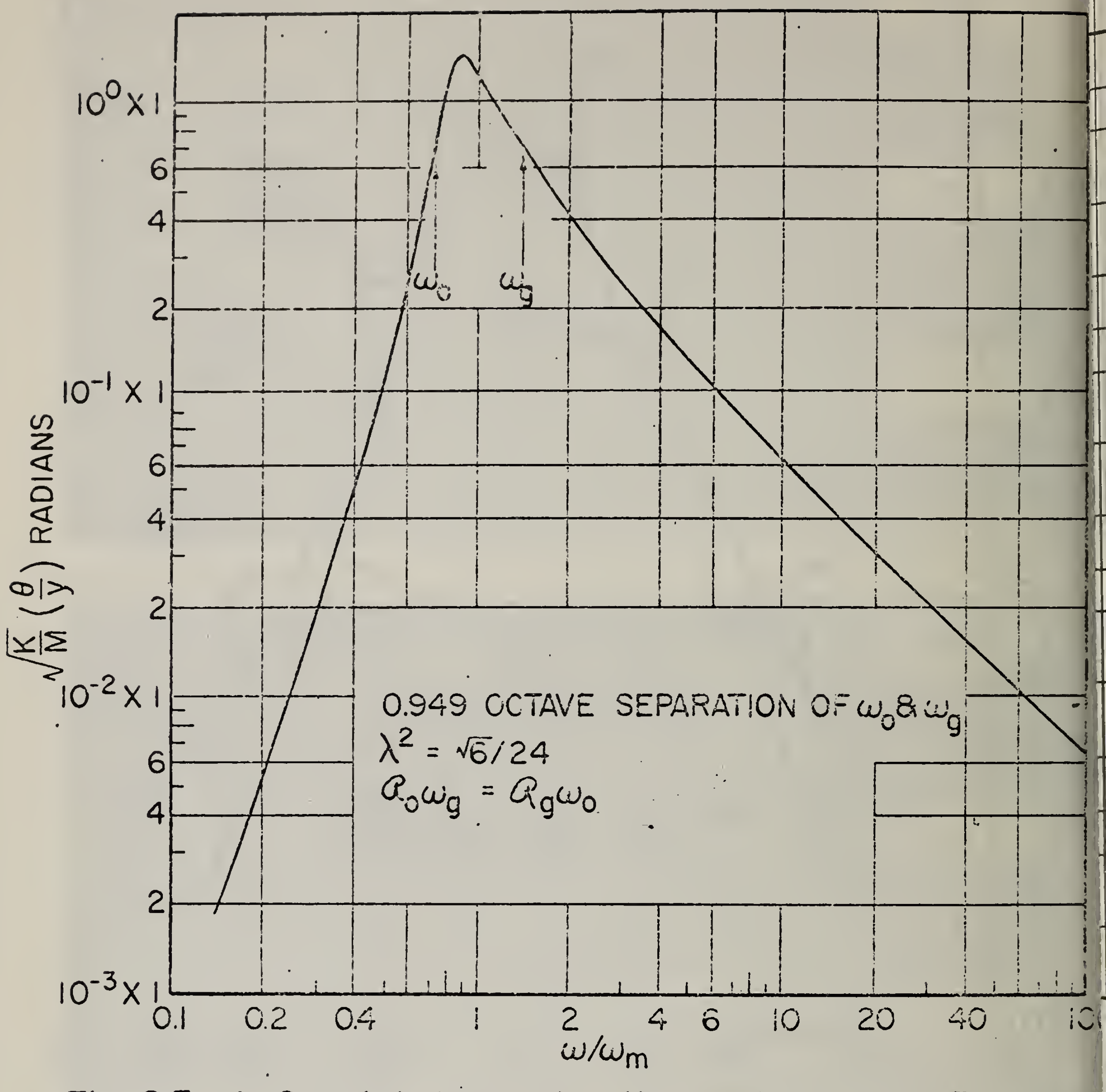


Fig. 6.3 A Special Case, Amplitude- Frequency Response Curve, Direct Coupled Instruments

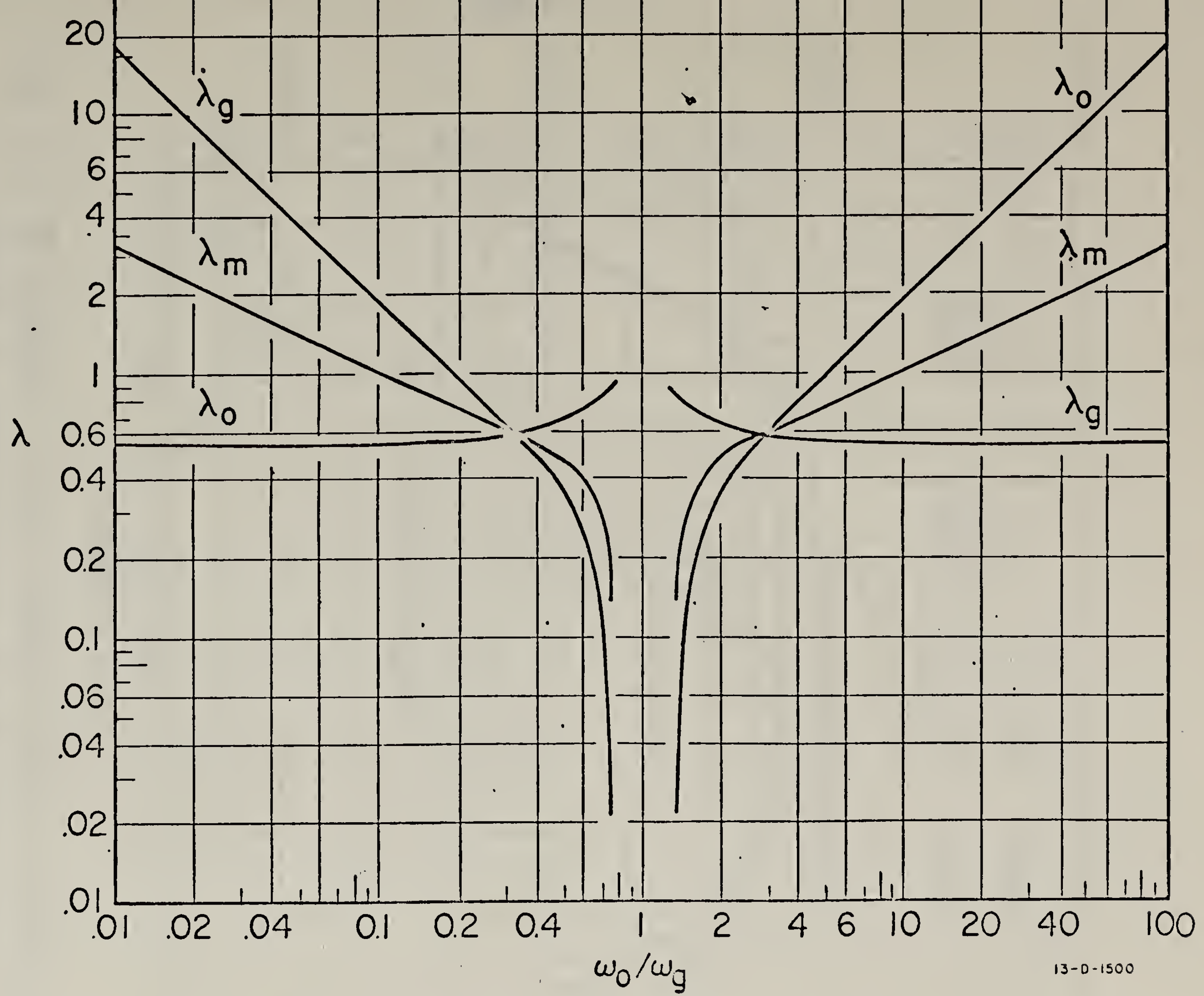


Fig. 6.4 Damping Required For Flat Amplitude With Direct Coupled Instruments

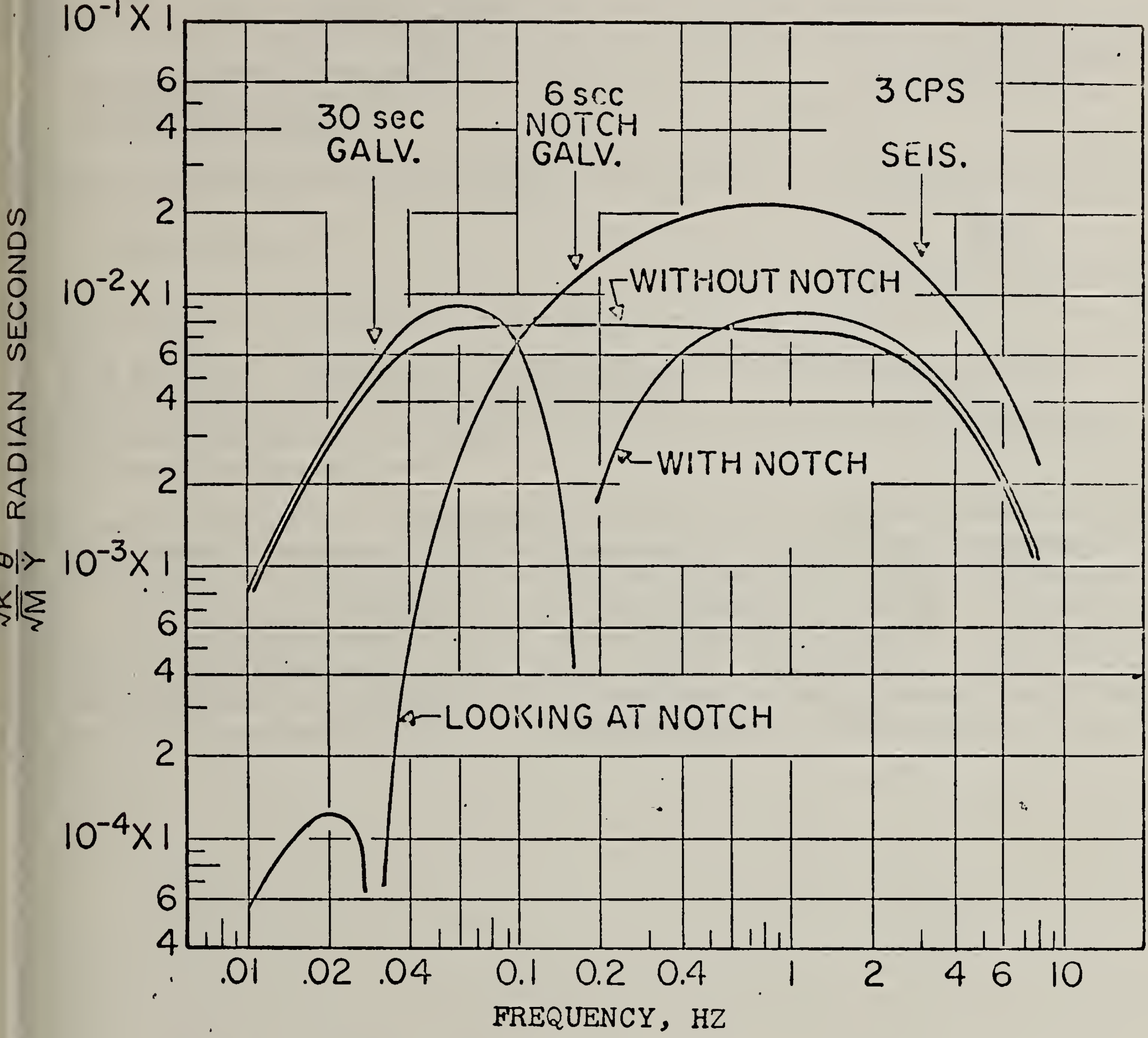


Fig. 7.3 A Possible Broad-Band Direct-Coupled System Using A Galvanometer As A Microseism Filter

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electromechanical transducer. If the inductance as well as the resistance of the
tem are included, the equations of motion are of the third order. These are
cussed in some detail. Response curves are developed for several seismometer-
vanometer combinations, some of which are suitable for telemetry. Techniques are
cussed for calibration, both in the laboratory and in the field. Practical and
oretical limitations are treated for circumstances in which thermal agitation is
ajor source of instrumental noise. The bibliography includes lists of equivalent
ations.

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lifier; seismometer; seismic system

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